Differential Equations (Math 285)

H1 We have considered the ODE $y' = -x/y$ as an example in the lecture. Actually there are four ODE's, viz. $y' = \pm x/y$ and $y' = \pm y/x$, which look very similar. Draw direction fields for the other three ODE's and determine their solutions in both implicit and explicit form (if possible).

H2 Determine all points $(t_0, y_0) \in \mathbb{R}^2$ such that there is a unique solution on $[t_0, \infty)$ of the IVP $y' = \sqrt{|y|}$, $y(t_0) = y_0$ ("the value at time t_0 determines the values at all future times $t > t_0$").

H3 Let $t_0, y_0, y_1 \in \mathbb{R}$. Show that the IVP

$$y'' = -y, \quad y(t_0) = y_0,\ y'(t_0) = y_1$$

has a unique solution.

H4 For each of the following ODE's, determine at least one nonzero solution by using the "Ansatz" $y(x) = a\,\mathrm{e}^{\alpha x}$ or $y(x) = b\,x^{\beta}$.

a) $y'' = y^2$;
b) $y'' - 5y' + 6y = 0$;
c) $y'' - 5y' + 6y = \mathrm{e}^x$;
d) $y'' - \frac{1}{2x}\,y' + \frac{1}{2x^2}\,y = 0$;
e) $(2x + 1)y'' + (4x - 2)y' - 8y = 0$;
f) $x^2(1 - x)y'' + 2x(2 - x)y' + 2(1 + x)y = 0$.

H5 Do two of the three exercises on the pendulum equation in [BDM17], Ch. 1.3 (Exercises 23–25 in the 11th global edition).

H6 *Optional Exercise*

Solve the functional differential equation $f' = f^{-1}$, where f^{-1} denotes the compositional inverse of f; i.e., $f\colon I \to I$ ($I \subseteq \mathbb{R}$ an interval) should be bijective, differentiable, and satisfy $f'\bigl(f(x)\bigr) = f\bigl(f'(x)\bigr) = x$ for all $x \in I$.

Due on Thu Jan 25, 9 am

The optional exercise can be handed in until Thu Feb 29, 9 am.

Instructions For your homework it is best to maintain 2 notebooks, which are handed in on alternate Fridays. You may also use A4 sheets, provided they are firmly stapled together.

Don't forget to write your name (English and Chinese) and your student ID on the first page.

Homework is handed in on Thursdays before the informal discussion session starts (late homework won't be accepted!) and will be returned on the next Thursday.

Answers to exercises must be justified; it is not sufficient to state only the final result of a computation.

Answers must be written in English.

For a full homework score it is sufficient to solve ca. 80 % of the mandatory homework exercises. Optional exercises contribute to the homework score, but they are usually more difficult and you should work on them only if you have sufficient spare time.

Solutions (prepared by TA's and TH)

1 (A) $y' = x/y$: Rewriting the ODE as $yy' - x = 0$ and integrating gives $\frac{1}{2}\,y^2 - \frac{1}{2}\,x^2 =$

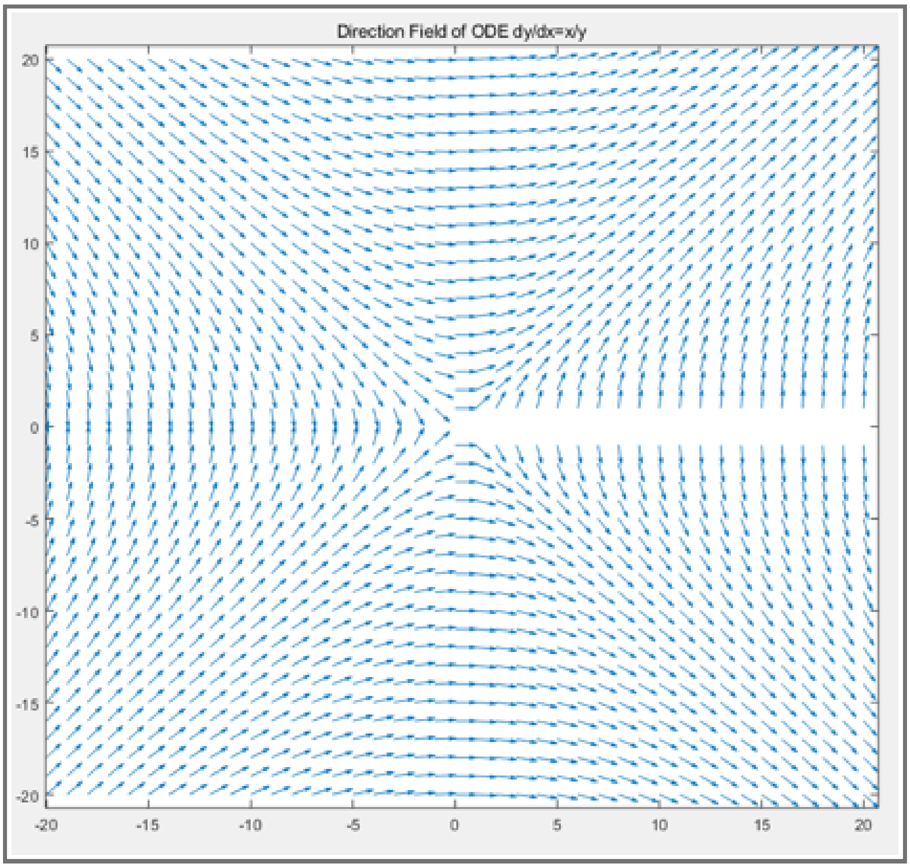


Figure 1: Direction field of $y' = x/y$

$C \in \mathbb{R}$. Replacing C by $C/2$ turns this into $y^2 - x^2 = C$ (implicit form), $y(x) = \pm\sqrt{x^2 + C}$ (explicit form). Explicit solutions have domain $\mathbb{R}$ if $C > 0$ and domains $\left(-\infty, -\sqrt{-C}\right)$, $\left(\sqrt{-C}, \infty\right)$ if $C \leq 0$. (For $C = 0$ the solutions, viz. $y(x) = \pm x$, formally are not defined at 0, since the domain of $y' = x/y$ excludes the x-axis.)

(B) $y' = y/x$: The solutions are $y(x) = Cx$, $C \in \mathbb{R}$, with domains $(-\infty, 0)$, $(0, \infty)$. Again the exclusion of $x = 0$ is artificial and due to the special form of the ODE. (It would be included if we rewrite the ODE as $xy' - y = 0$.) That all solutions have been found, follows from $(y/x)' = (xy' - y)/x^2 = 0$, which implies $y/x = C$ is a constant.

(C) $y' = -y/x$: The solutions are $y(x) = C/x$, $C \in \mathbb{R}$, with domains $(-\infty, 0)$, $(0, \infty)$. That these are all solutions follows from $(xy)' = y + xy' = 0$, which implies $xy = C$ is a constant.

2 As discussed in the lecture, the solutions of $y' = \sqrt{|y|}$ form the 2-parameter family

$$y(t) = \begin{cases} -\frac{1}{4}(t - c_1)^2 & \text{if } t < c_1, \\ 0 & \text{if } c_1 \leq t \leq c_2, \\ \frac{1}{4}(t - c_2)^2 & \text{if } t > c_2. \end{cases}$$

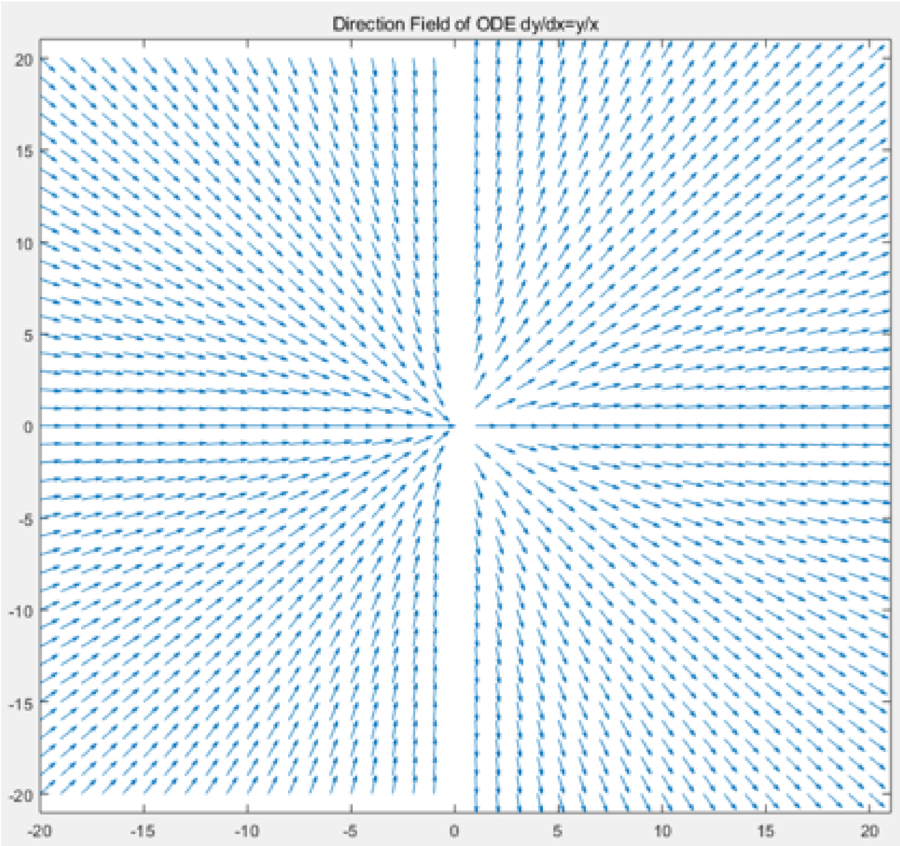


Figure 2: Direction field of $y' = y/x$

One or both of c_1, c_2 may be infinite ($c_1 = -\infty$, $c_2 = \infty$).
Claim: Solutions are uniquely determined for all $t \geq t_0$ iff $y_0 = y(t_0) > 0$.
If $y_0 > 0$ then near t_0 the solution must have the form $y(t) = \frac{1}{4}(t - c_2)^2$ with c_2 determined from $\frac{1}{4}(t_0 - c_2)^2 = y_0$, i.e, $t_0 - c_2 = 2\sqrt{y_0}$ and $c_2 = t_0 - 2\sqrt{y_0}$. (Note that c_2 must be smaller than t_0 in this case.) Thus $y(t)$ is determined for all $t \geq t_0$.
If $y_0 = 0$ then $y(t)$ is not uniquely determined, since $y_1(t) = 0$ for $t \geq t_0$ and $y_2(t) = \frac{1}{4}(t - t_0)^2$ for $t \geq t_0$ are two different solutions.
If $y_0 < 0$, then near t_0 the solution has the form $y(t) = -\frac{1}{4}(t - c_1)^2$ form some $c_1 > t_0$ (determined as in the case $y_0 > 0$), and hence $y(c_1) = 0$. Thus we are back in the previous case and the solution is not unique.

3 If $y(t)$ (with domain $\mathbb{R}$) solves the given IVP then $z(t) = y(t + t_0)$ solves the IVP $z'' = -z$, $z(0) = y_0$, $z'(0) = y_1$. From Example 10 in the lecture we know that the unique solution of this IVP is $z(t) = y_0 \cos t + y_1 \sin t$. Hence

$$y(t) = z(t - t_0) = y_0 \cos(t - t_0) + y_1 \sin(t - t_0)$$

is unique as well.

4 a) $y(x) = bx^\beta \Longrightarrow y''(x) = b\beta(\beta - 1)x^{\beta-2} \doteq b^2x^{2\beta}$
The only nonzero solution is $\beta = -2$, $b = 6$, i.e., $y(x) = 6x^{-2}$. (For this we use that $b_1x^{\beta_1} = b_2x^{\beta_2}$ holds for all x in an intervall of positive length iff $b_1 = b_2 \wedge \beta_1 = \beta_2$, provided that both b_1, b_2 are nonzero.

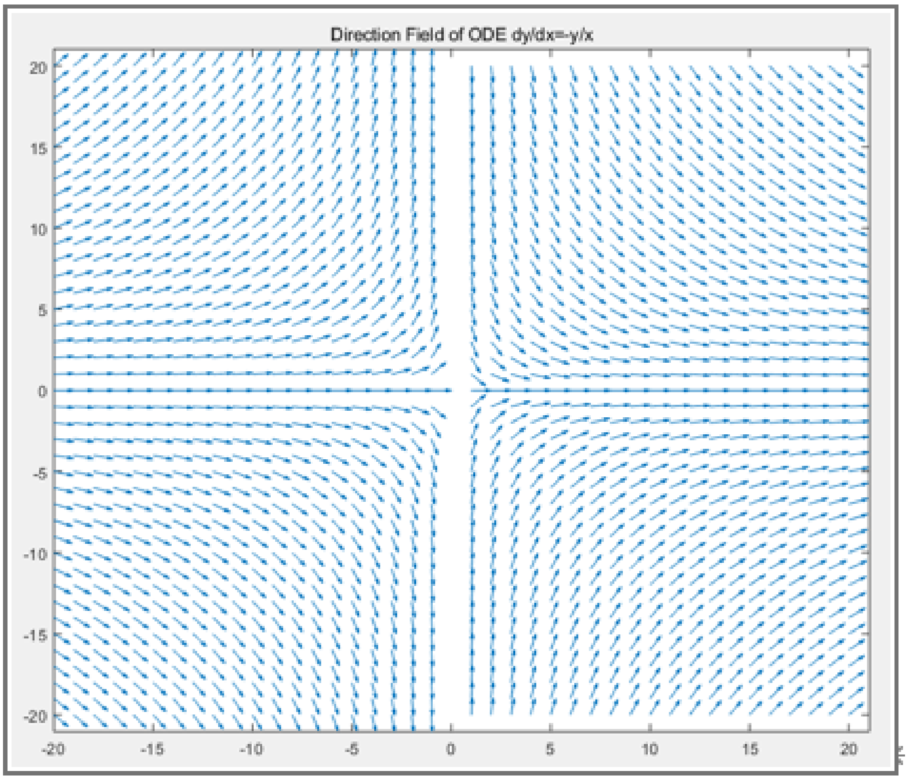


Figure 3: Direction field of $y' = -y/x$

The other "Ansatz" doesn't work, because $\mathrm{e}^{\alpha x}$ and $(\mathrm{e}^{\alpha x})^2 = \mathrm{e}^{2\alpha x}$ aren't scalar multiples of each other if $\alpha \neq 0$.

b) The Ansatz $y(x) = b\,x^\beta$ doesn't work, because in this case $y'' - 5y' + 6y$ involves x^β, $x^{\beta-1}$, $x^{\beta-2}$, which don't cancel.

For $y(x) = a\mathrm{e}^{\alpha x}$ we obtain

$$y'' - 5y' + 6y = \alpha^2 a\mathrm{e}^{\alpha x} - 5\alpha a\mathrm{e}^{\alpha x} + 6a\mathrm{e}^{\alpha x} = (\alpha^2 - 5\alpha + 6)a\mathrm{e}^{\alpha x},$$

which can be made zero by taking α as a root of $X^2 - 5X + 6$, i.e., $\alpha \in \{2, 3\}$. For $a = 1$ this gives the two solutions $y_1(x) = \mathrm{e}^{2x}$, $y_2(x) = \mathrm{e}^{3x}$. Every linear combination $y(x) = a_1\,\mathrm{e}^{2x} + a_2\,\mathrm{e}^{3x}$, $a_1, a_2 \in \mathbb{R}$, is then a solution as well.

c) Again it is clear that $y(x) = b\,x^\beta$ doesn't work. For $y(x) = a\mathrm{e}^{\alpha x}$ we see from b) that we need to solve

$$(\alpha^2 - 5\alpha + 6)a\mathrm{e}^{\alpha x} = \mathrm{e}^x.$$

This can be done: Set $\alpha = 1$ and solve $(\alpha^2 - 5\alpha + 6)a = 2a = 1$ for a, i.e., $a = 1/2$. A solution is therefore $y(x) = \frac{1}{2}\,\mathrm{e}^x$.

d) Clearly only the 2nd Ansatz can work, and in fact it does (w.l.o.g. set $b = 1$):

$$y'' - \frac{1}{2x}\,y' + \frac{1}{2x^2}\,y = \beta(\beta-1)x^{\beta-2} - \frac{\beta x^{\beta-1}}{2x} + \frac{x^\beta}{2x^2} = \left(\beta^2 - \frac{3}{2}\beta + \frac{1}{2}\right)x^{\beta-2} = 0$$

if β is a root of $X^2 - \frac{3}{2}X + \frac{1}{2}$, i.e., $\beta \in \{1, \frac{1}{2}\}$. Thus $y_1(x) = x$, $y_2(x) = \sqrt{x}$ are two solutions, and, more generally, $y(x) = a_1 x + a_2\sqrt{x}$, $a_1, a_2, \in \mathbb{R}$ are solutions.

e) Here $y(x) = \mathrm{e}^{\alpha x}$ works (the constant a is arbitrary and can be set to 1):

$$(2x+1)\alpha^2\mathrm{e}^{\alpha x} + (4x-2)\alpha\mathrm{e}^{\alpha x} - 8\mathrm{e}^{\alpha x} = \big((2x+1)\alpha^2 + (4x-2)\alpha - 8\big)\mathrm{e}^{\alpha x} = 0.$$

Since $\mathrm{e}^{\alpha x} \neq 0$, the polynomial

$$(2x+1)\alpha^2 + (4x-2)\alpha - 8 = (2\alpha^2 + 4\alpha)x + \alpha^2 - 2\alpha - 8$$

must be zero, since it vanishes for infinitely many x, and hence

$$2\alpha^2 + 4\alpha = \alpha^2 - 2\alpha - 8 = 0.$$

The unique solution is $\alpha = -2$, showing that $y(x) = \mathrm{e}^{-2x}$ solves the given ODE.

f) Here $y(x) = x^\beta$ works (the constant b is arbitrary and can be set to 1):

$$\begin{aligned} &x^2(1-x)\beta(\beta-1)x^{\beta-2} + 2x(2-x)\beta x^{\beta-1} + 2(1+x)x^\beta \\ &\qquad = \big(x^2(1-x)\beta(\beta-1) + 2x(2-x)\beta x + 2(1+x)x^2\big)x^{\beta-2} = 0. \end{aligned}$$

$\Longrightarrow$ The first factor, which is

$$(-\beta^2 - \beta + 2)x^3 + (\beta^2 + 3\beta + 2)x^2 = -(\beta-1)(\beta+2)x^3 + (\beta+1)(\beta+2)x^2,$$

must be the zero polynomial. The unique solution is $\beta = -2$, giving the solution $y(x) = x^{-2}$ of the ODE.

5 Ex. 23 (a) The relation between angular, angular velocity and linear velocity is:

$$v = \omega R = R\frac{d\theta}{dt}$$

Therefore, the kinetic energy T can be represented as:

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m(R\frac{d\theta}{dt})^2 = \frac{1}{2}mL^2\frac{d\theta}{dt}$$

(b) The potential energy V of the pendulum can be represented as:

$$V = mgh = mg(L - Lcos\theta) = mgL(1 - cos\theta)$$

(c) The total energy E can be represented as:

$$E = T + V = \frac{1}{2}mL^2(\frac{d\theta}{dt})^2 + mgL(1 - cos\theta)$$

Hence,

$$\frac{dE}{dt} = \frac{1}{2}mL^2(2\frac{d\theta}{dt})\frac{d^2\theta}{dt^2} - mgL(-sin\theta)\frac{d\theta}{dt}$$

The total energy E is invariant with time. Therefore,

$$\frac{dE}{dt} = \frac{1}{2}mL^2(2\frac{d\theta}{dt})\frac{d^2\theta}{dt^2} - mgL(-sin\theta)\frac{d\theta}{dt} = 0$$

Simplify the above equation:

$$\frac{d^2\theta}{dt^2} + \frac{g}{L} sin\theta = 0$$

Ex. 24

(a) According to the definition of angular momentum, it can be represented as:

$$M = L(mv) = mL(R\frac{d\theta}{dt}) = mL^2\frac{d\theta}{dt}$$

(b) According to the principle of angular momentum conservation,

$$\frac{dM}{dt} = -(mgsin\theta)L$$

Therefore,

$$\frac{dM}{dt} = mL^2\frac{d^2\theta}{dt^2} = -(mgsin\theta)L$$

Simplify the above equation:

$$\frac{d^2\theta}{dt^2} + \frac{g}{L} sin\theta = 0$$

Ex. 25 (a) The free-body diagram is shown below:

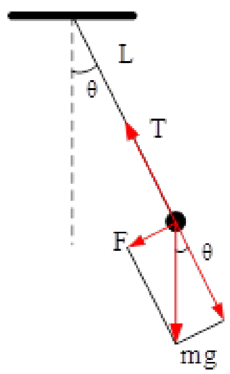


Figure 4: pendulum

(b)(c) Apply Newton's Law of motion in the direction tangential to the circular arc.

$$F = mgsin\theta = ma$$

$$a = -\frac{dv}{dt} = -\frac{d}{dt}(\omega L) = -L\frac{d\omega}{dt} = -L\frac{d^2\theta}{dt^2}$$

Therefore,

$$F = mgsin\theta = -m(L\frac{d^2\theta}{dt^2})$$

Simplify the above equation:

$$\frac{d^2\theta}{dt^2} + \frac{g}{L} sin\theta = 0$$

6 Since the derivative and compositional inverse of a power function is again a power function, the power function „Ansatz" $f(x) = bx^\beta$ of H4 is reasonable. Since $y = bx^\beta \iff x = (y/b)^{1/\beta}$, we obtain the condition

$$b\beta x^{\beta-1} = f'(x) = f^{-1}(x) = (x/b)^{1/\beta} = b^{-1/\beta}x^{1/\beta}$$

for such a solution. This can only hold if $\beta - 1 = 1/\beta$ and $b\beta = b^{-1/\beta}$. Thus β should satisfy the quadratic equation $\beta^2 - \beta - 1 = 0$, i.e., $\beta = \frac{1\pm\sqrt{5}}{2}$, and b should satisfy $b^{1+1/\beta}\beta = b^\beta\beta = 1$. If β is negative, there is no real solution. For the positive root $\beta = \frac{1+\sqrt{5}}{2}$ we obtain $b = (1/\beta)^{1/\beta}$ and hence the solution

$$f_1(x) = \left(\frac{-1+\sqrt{5}}{2}\right)^{\frac{-1+\sqrt{5}}{2}} x^{\frac{1+\sqrt{5}}{2}}, \quad x \in [0, +\infty).$$

Since $\beta > 1$, f_1 is differentiable at $x = 0$ with $f_1(0) = f_1'(0) = 0$.
There is a second solution on $(-\infty, 0)$, which can be obtained using the Ansatz $f(x) = b(-x)^\beta$. Here $f'(x) = -b\beta(-x)^{\beta-1}$, $f^{-1}(x) = -(x/b)^{1/\beta} = -(-x/-b)^{1/\beta} = -(-b)^{-1/\beta}(-x)^{1/\beta}$, (requiring $b < 0$), so that again $\beta^2 - \beta - 1 = 0$, and $(-b)^\beta\beta = -1$. Now $\beta = \frac{1-\sqrt{5}}{2}$ must be the negative root, and $b = -(-1/\beta)^{1/\beta}$. This gives the solution

$$f_2(x) = -\left(\frac{1+\sqrt{5}}{2}\right)^{\frac{-1-\sqrt{5}}{2}} (-x)^{\frac{1-\sqrt{5}}{2}}, \quad x \in (-\infty, 0).$$

According to the available literature the solution f_1 on $[0, +\infty)$ is unique, but I do not know this for f_2 and also whether there are further solutions with other domains.

Differential Equations (Math 285)

H7 Solve the initial value problem $y' + 4y = 8e^{-4t} + 20$, $y(0) = 0$ and determine $y_\infty = \lim_{t\to\infty} y(t)$ for the solution.

H8 Solve $y' - 2y = e^{ct}$, $y(0) = 1$ and graph the solution for

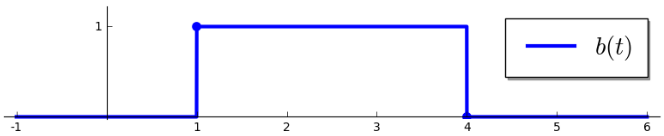
a) $c = 2$; b) $c = 2.01$.

What do you observe?

H9 The Heaviside function $\mathrm{u}\colon \mathbb{R} \to \mathbb{R}$ is defined by

$$\mathrm{u}(t) = \begin{cases} 0 & \text{if } t < 0, \\ 1 & \text{if } t \geq 0. \end{cases}$$

Express $b(t)$ (cf. picture) in terms of $\mathrm{u}(t)$, solve the initial value problem $y' + 2y = b(t)$, $y(0) = 0$, and determine y_∞ (cf. H7).



H10 a) Write the following complex numbers in polar form:

(i) $\sqrt{3}\,\mathrm{i} + 1$; (ii) $\sqrt{3}\,\mathrm{i} - 1$; (iii) $\mathrm{i} - \sqrt{3}$.

b) Determine the general solution of the following ODE's:

(i) $y' + y = \cos\left(\sqrt{3}\,t\right)$; (ii) $y' - y = \cos\left(\sqrt{3}\,t\right)$;

(iii) $y' - \sqrt{3}\,y = \cos t + \sin t$.

c) Suppose $A\colon I \to \mathbb{C}$, $t \mapsto A_1(t) + \mathrm{i}\,A_2(t)$ is differentiable (i.e., $A_1 = \operatorname{Re} A$ and $A_2 = \operatorname{Im} A$ are differentiable). Show that $I \to \mathbb{C}$, $t \mapsto e^{A(t)}$ is differentiable as well, and

$$\frac{\mathrm{d}}{\mathrm{d}t} e^{A(t)} = A'(t) e^{A(t)}.$$

Hint: Start with $e^{A(t)} = e^{A_1(t) + \mathrm{i}\,A_2(t)} = e^{A_1(t)} e^{\mathrm{i}\,A_2(t)} = e^{A_1(t)} \cos A_2(t) + \mathrm{i}\, e^{A_1(t)} \sin A_2(t)$.

H11 a) Show that in the 3rd model $mv' = mg - kv^2$ for a falling object released at height s_0 the terminal velocity v_T of the object at time of impact is given by

$$v_T = \sqrt{\frac{mg}{k}} \cdot \sqrt{1 - e^{-2ks_0/m}}.$$

Hint: Consider the velocity as a function $v(s)$ of the distance s traveled. Show that $y(s) = v(s)^2$ satisfies the ODE $my' = 2mg - 2ky$.

b) The limiting velocity of a falling basketball ($m = 620\,\mathrm{g}$) has been estimated at $20\,\mathrm{m/s}$. Using this data, graph v_T as a function of s_0. For which heights s_0 does the basketball reach $50\,\%$, $90\,\%$, and $99\,\%$ of its limiting velocity?

H12 a) Let $f_\lambda(t) = \mathrm{e}^{\lambda t}$ for $\lambda \in \mathbb{R}$. Show that $\{f_\lambda; \lambda \in \mathbb{R}\}$ is linearly independent in $\mathbb{R}^\mathbb{R}$.

Hint: Suppose there exists $r \in \mathbb{Z}^+$ and distinct numbers $\lambda_1, \dots, \lambda_r, c_1, \dots, c_r \in \mathbb{R}$ such that

$$c_1\mathrm{e}^{\lambda_1 t} + c_2\mathrm{e}^{\lambda_2 t} + \cdots + c_r\mathrm{e}^{\lambda_r t} = 0 \quad \text{for all } t \in \mathbb{R}. \tag{$\star$}$$

Assuming $\lambda_1 < \lambda_2 < \cdots < \lambda_r$ and $c_r \neq 0$, divide this equation by $\mathrm{e}^{\lambda_r t}$ and let $t \to +\infty$ to obtain a contradiction.

b) For $\lambda \in \mathbb{C}$ the functions $f_\lambda(t) = \mathrm{e}^{\lambda t}$ belong to the vector space $\mathbb{C}^\mathbb{R}$ of all complex-valued functions on $\mathbb{R}$ (with scalar multiplication by complex numbers). Show that $\{f_\lambda; \lambda \in \mathbb{C}\}$ is linearly independent in $\mathbb{C}^\mathbb{R}$.

Hint: The proof outlined in a) breaks down in the complex case. Instead differentiate the identity in $(\star)$ j times, $0 \leq j < r$, and set $t = 0$.

c) Let $c_\lambda(t) = \cos(\lambda t)$, $s_\lambda(t) = \sin(\lambda t)$. Show that $\{c_\lambda; \lambda \in \mathbb{R}, \lambda \geq 0\} \cup \{s_\lambda; \lambda \in \mathbb{R}, \lambda > 0\}$ is linearly independent in $\mathbb{R}^\mathbb{R}$.

Due on Thu Feb 29, 9 am

Exercises H12 b) and H12 c) are optional.

Solutions

7 According to the particular solution formula,

$$\begin{aligned} y_p(t) &= e^{-4t}\int_0^t (8e^{-4s}+20)e^{4s}\,ds \\ &= (8t-5)e^{-4t}+5 \end{aligned}$$

$$\begin{aligned} \implies y(t) &= Ce^{-4t}+y_p(t) \\ &= Ce^{-4t}+(8t-5)e^{-4t}+5, \quad C\in\mathbb{R}. \end{aligned}$$

Plug the initial condition $y(0)=0$ into the general solution:

$$y(0) = Ce^{-4*0}+(8*0-5)e^{-4*0}+5 = 0$$

$$\implies C = 0$$

$$\implies y(t) = (8t-5)e^{-4t}+5$$

$$y_\infty = \lim_{t\to\infty}\left[(8t-5)e^{-4t}+5\right] = 5$$

. (It can also be seen directly that the particular solution $y_p(t)$ satisfies already $y_p(0)=0$.)

8 a)

$$\because c = 2$$

$$\therefore y' = 2y+e^{2t}$$

According to the particular solution formula,

$$y_p(t) = e^{2t}\int_0^t e^{2s}e^{-2s}\,ds = te^{2t}$$

$$y(t) = te^{2t}+C_1e^{2t}$$

Plug the initial condition $y(0)=1$ into the general solution

$$y(0) = 0*e^{2*0}+C_1e^{2*0} = 1$$

$$\implies C_1 = 1$$

$$\implies y(t) = (t+1)e^{2t}$$

b)

$$\because c = 2.01$$

$$\therefore y' = 2y+e^{2.01t}$$

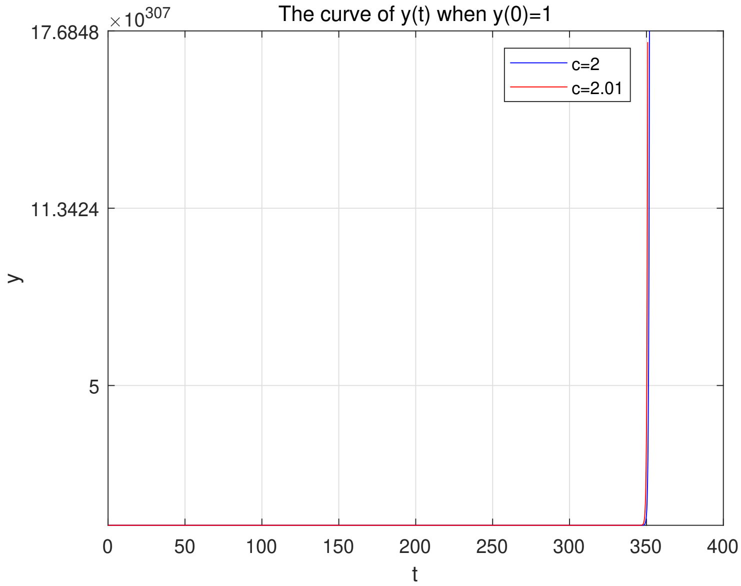
According to the particular solution formula,

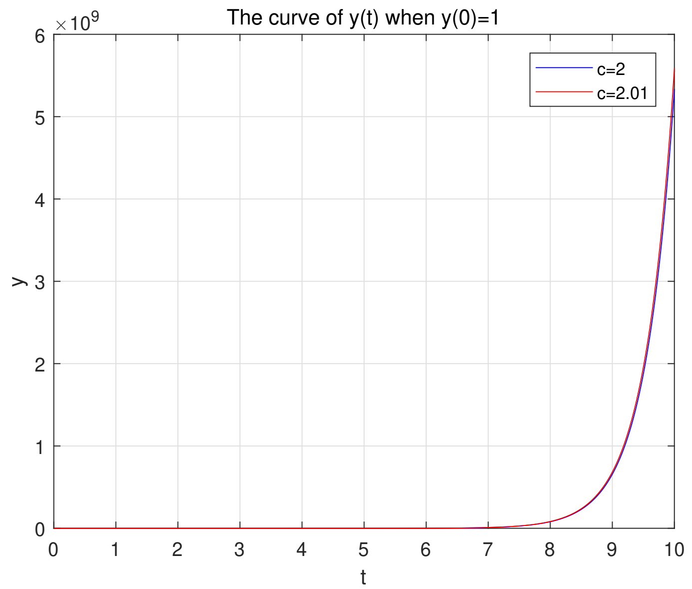
$$y_p(t) = e^{2t}\int_0^t e^{2.01s}e^{-2s}\,ds = 100e^{2.01t}-100e^{2t}$$

$$y(t) = 100e^{2.01t}+(C_2-100)e^{2t}$$

Plug the initial condition $y(0) = 1$ into the general solution

$$y(0) = 100e^{2.01*0} + (C_2 - 100)e^{2*0} = 1$$
$$\implies C_2 = 1$$
$$\implies y(t) = 100e^{2.01t} - 99e^{2t}$$





Therefore, there is no significant difference between these two functions, provided t is not too large. On the other hand, we have

$$\frac{100\,\mathrm{e}^{2.01\,t} - 99\,\mathrm{e}^{2t}}{(t+1)\mathrm{e}^{2t}} = \frac{100\,\mathrm{e}^{0.01\,t} - 99}{(t+1)},$$

and the quotient grows exponentially. Hence for large t the solution of b) is significantly larger.

9 We have $b(t) = \mathrm{u}(t-1) - \mathrm{u}(t-4)$ for $t \in \mathbb{R}$ (this also holds at $t = 1$ and $t = 4$).

$$y' = -2y + b(t)$$
$$b(t) = \begin{cases} 0, & \text{if } t < 1 \text{ or } t \geq 4 \\ 1, & \text{if } 1 \leq t < 4 \end{cases}$$

When $t < 1$, $b(t) = 0$
$\Longrightarrow$ The equation is $y' + 2y = 0$, which is homogeneous.

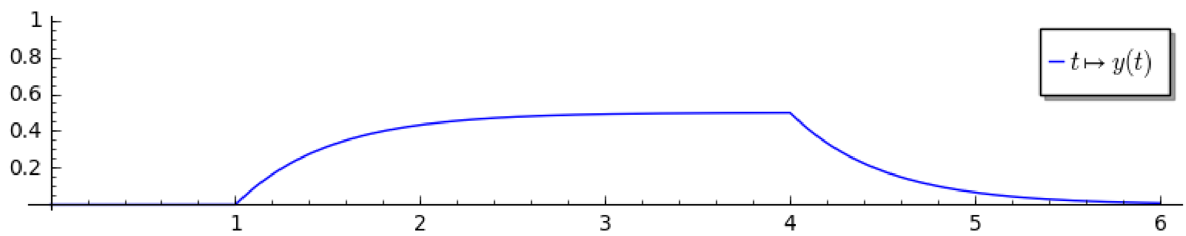
$$\therefore y(t) = C_1 e^{-2t}, C_1 \in \mathbb{R}$$
$$y(0) = C_1 * e^0 = 0 \Longrightarrow C_1 = 0$$
$$\therefore y(t) = 0$$

When $1 \leq t < 4$, $b(t) = 1$
$\Longrightarrow$ The equation is $y' + 2y = 1$, which is inhomogeneous and has $y_p(t) = \frac{1}{2}$ as particular solution.

$$\therefore y(t) = C_2 e^{-2t} + \frac{1}{2}, C_2 \in \mathbb{R}$$
$$y(1) = C_2 e^{-2*1} + \frac{1}{2} = 0 \Longrightarrow C_2 = -\frac{1}{2}e^2 \qquad \text{(Continuity at } t = 1)$$
$$\therefore y(t) = -\frac{1}{2}e^{2-2t} + \frac{1}{2}$$

When $t \geq 4$, $b(t) = 0$
$\Longrightarrow$ The equation is again $y' + 2y = 0$.

$$\therefore y(t) = C_3 e^{-2t}, C_3 \in \mathbb{R}$$
$$y(4) = C_3 e^{-2*4} = \frac{1}{2}(1 - e^{-6}) \Longrightarrow C_3 = \frac{1}{2}(e^8 - e^2) \qquad \text{(Continuity at } t = 4)$$
$$\therefore y(t) = \frac{1}{2}(e^8 - e^2)e^{-2t}$$
$$y_\infty = \lim_{t\to\infty} y(t) = \lim_{t\to\infty} \frac{1}{2}(e^8 - e^2)e^{-2t} = 0$$



10 a) (i) $\sqrt{3}\,i + 1 = \sqrt{3+1}\, e^{i \arctan(\sqrt{3})} = 2\, e^{i\frac{\pi}{3}}$

(ii) $\sqrt{3}\,i - 1 = \sqrt{3+1}\, e^{i(\pi + \arctan(-\sqrt{3}))} = 2\, e^{i\frac{2\pi}{3}}$

(iii) $i - \sqrt{3} = \sqrt{3+1}\, e^{[\pi + i \arctan(-\frac{\sqrt{3}}{3})]} = 2\, e^{i\frac{5\pi}{6}}$

b) (i) Complexifying this ODE leads to $z' = -z + e^{i\sqrt{3}\,t}$. If $z(t)$ solves the complex ODE, $y_p(t) = \operatorname{Re} z(t)$ will be a particular solution of $y' + y = \cos(\sqrt{3}\,t)$.
Since $(e^{i\sqrt{3}\,t})' = i\sqrt{3}\,e^{i\sqrt{3}\,t}$, it is reasonable to guess that there exists a particular solution of the form $z(t) = C\,e^{i\sqrt{3}\,t}$ with $C \in \mathbb{C}$.

$$z'(t) = Ci\sqrt{3}e^{i\sqrt{3}t} = -(Ce^{i\sqrt{3}t}) + e^{i\sqrt{3}t} \iff Ci\sqrt{3} = 1 - C \iff C = \frac{1}{1+i\sqrt{3}}$$

$$\implies z(t) = \frac{1-i\sqrt{3}}{4}(\cos(\sqrt{3}\,t) + i\sin(\sqrt{3}\,t))$$

$$\therefore y_p(t) = \operatorname{Re} z(t) = \frac{1}{4}\cos(\sqrt{3}t) + \frac{\sqrt{3}}{4}\sin(\sqrt{3}t)$$

So the general (real) solution of $y' + y = \cos(\sqrt{3}\,t)$ is

$$y(t) = C_1 e^{-t} + \frac{1}{4}\cos(\sqrt{3}\,t) + \frac{\sqrt{3}}{4}\sin(\sqrt{3}\,t), \quad C_1 \in \mathbb{R}.$$

(ii) Here we have $z' = z + e^{i\sqrt{3}\,t}$ and can use the same “Ansatz” as in (i).

$$z'(t) = Ci\sqrt{3}e^{i\sqrt{3}t} = (Ce^{i\sqrt{3}t}) + e^{i\sqrt{3}t} \iff Ci\sqrt{3} = 1 + C \iff C = \frac{1}{-1+i\sqrt{3}}$$

$$\implies z(t) = -\frac{1+i\sqrt{3}}{4}(\cos(\sqrt{3}t) + i\sin(\sqrt{3}t))$$

$$\therefore y_p(t) = \operatorname{Re} z(t) = -\frac{1}{4}\cos(\sqrt{3}t) + \frac{\sqrt{3}}{4}\sin(\sqrt{3}t)$$

So the general solution of $y' - y = \cos(\sqrt{3}\,t)$ is

$$y(t) = C_1 e^{t} - \frac{1}{4}\cos(\sqrt{3}\,t) + \frac{\sqrt{3}}{4}\sin(\sqrt{3}\,t), \quad C_1 \in \mathbb{R}.$$

(iii)

$$y' - \sqrt{3}\,y = \cos t + \sin t \iff y' = \sqrt{3}\,y + \sqrt{2}\sin\left(\frac{\pi}{4} + t\right)$$

Complexifying this ODE leads to $z' = \sqrt{3}\,z + \sqrt{2}\,e^{i(\frac{\pi}{4}+t)} = \sqrt{3}\,z + (1+i)\mathrm{e}^{it}$ and, since we have complexified a sine, this time $y_p(t) = \operatorname{Im} z(t)$ will give a particular solution of the original ODE. Using the “Ansatz” $z(t) = C\,\mathrm{e}^{it}$, $C \in \mathbb{C}$, we obtain

$$z'(t) = Ci\,\mathrm{e}^{it} = \sqrt{3}\,C\,\mathrm{e}^{it} + (1+i)\mathrm{e}^{it} \iff Ci = \sqrt{3}\,C + 1 + i$$

$$\iff C = \frac{1+i}{-\sqrt{3}+i} = \frac{(1+i)(-\sqrt{3}-i)}{4} = \frac{1-\sqrt{3}}{4} - \frac{1+\sqrt{3}}{4}i$$

$$\begin{aligned}\implies y_p(t) &= \operatorname{Im}\left[\left(\frac{1-\sqrt{3}}{4} - \frac{1+\sqrt{3}}{4}i\right)\mathrm{e}^{it}\right] \\ &= -\frac{1+\sqrt{3}}{4}\cos t + \frac{1-\sqrt{3}}{4}\sin t \\ &= -\frac{\sqrt{2}}{4}\cos\left(\frac{\pi}{4} + t\right) - \frac{\sqrt{6}}{4}\sin\left(\frac{\pi}{4} + t\right)\end{aligned}$$

So the general solution of $y' - \sqrt{3}\,y = \cos t + \sin t$ is

$$\begin{aligned} y(t) &= C_1 e^{\sqrt{3}\,t} - \frac{1+\sqrt{3}}{4}\cos t + \frac{1-\sqrt{3}}{4}\sin t \\ &= C_1 e^{\sqrt{3}\,t} - \frac{\sqrt{2}}{4}\cos\left(\frac{\pi}{4}+t\right) - \frac{\sqrt{6}}{4}\sin\left(\frac{\pi}{4}+t\right), \quad C_1 \in \mathbb{R}. \end{aligned}$$

c) Using the rules for differentiating real-valued functions (in particular, the rule $\frac{\mathrm{d}}{\mathrm{d}t}\, e^{A_1(t)} = A_1'(t)\mathrm{e}^{A_1(t)}$, which is an instance of the chain rule), we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left[\mathrm{e}^{A_1(t)}\cos A_2(t)\right] &= A_1'(t)\mathrm{e}^{A_1(t)}\cos A_2(t) - \mathrm{e}^{A_1(t)}\sin A_2(t)\, A_2'(t), \\ \frac{\mathrm{d}}{\mathrm{d}t}\left[\mathrm{e}^{A_1(t)}\sin A_2(t)\right] &= A_1'(t)\mathrm{e}^{A_1(t)}\sin A_2(t) + \mathrm{e}^{A_1(t)}\cos A_2(t)\, A_2'(t), \end{aligned}$$

and hence

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{A(t)} &= \left(A_1'(t)\mathrm{e}^{A_1(t)}\cos A_2(t) - \mathrm{e}^{A_1(t)}\sin A_2(t)A_2'(t)\right) + \mathrm{i}\left(A_1'(t)\mathrm{e}^{A_1(t)}\sin A_2(t) + \mathrm{e}^{A_1(t)}\cos A_2(t)A_2'(t)\right) \\ &= \left(A_1'(t) + \mathrm{i}\,A_2'(t)\right)\left(\mathrm{e}^{A_1(t)}\cos A_2(t) + \mathrm{i}\,\mathrm{e}^{A_1(t)}\sin A_2(t)\right) \\ &= A'(t)\mathrm{e}^{A(t)}. \end{aligned}$$

Of course, this also proves that $t \mapsto \mathrm{e}^{A(t)}$ is differentiable.

11 a)

$$mv' = mg - kv^2 \Longleftrightarrow m\frac{\mathrm{d}v}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}t} = mg - kv^2 \Longleftrightarrow mv\frac{\mathrm{d}v}{\mathrm{d}s} = mg - kv^2$$

Assuming $y(s) = v(s)^2$, we have $\frac{\mathrm{d}y}{\mathrm{d}s} = 2v\,\frac{\mathrm{d}v}{\mathrm{d}s}$.
By substituting this into the equation $mv\,\frac{\mathrm{d}v}{\mathrm{d}s} = mg - kv^2$, we get

$$my' = 2mg - 2ky$$

$$\Longrightarrow y' = -\frac{2k}{m}y + 2g.$$

(*Note added on Mar 2, 2024:* Students have pointed out that the above computation is confusing, because v' means $v'(t)$ and not $v'(s)$, and y' means $y'(s)$. We can avoid this by writing $m\,\frac{\mathrm{d}v}{\mathrm{d}t} = mg - kv^2$ instead of $mv' = mg - kv^2$ and $m\,\frac{\mathrm{d}y}{\mathrm{d}s} = 2mg - 2ky$ further down. While it is custom in applied mathematics to use $v(t)$, $v(s)$ simultaneously, you should be aware that given $v(t)$ the second is a derived notion, actually meaning $s \mapsto v(t(s))$, where $t(s)$ denotes the inverse function of $s(t)$. It is "speed parametrized with respect to arc length", so-to-speak.)

According to the general solution formula,

$$y(s) = Ce^{-\frac{2k}{m}s} + e^{-\frac{2k}{m}s}\int_0^s 2ge^{\frac{2k}{m}\lambda}\,d\lambda = (C - \frac{mg}{k})e^{-\frac{2k}{m}s} + \frac{mg}{k}$$

$$\because v(0) = 0$$

$$\therefore y(0) = C - \frac{mg}{k} + \frac{mg}{k} = 0 \Longrightarrow C = 0$$

$$\Longrightarrow y = \frac{mg}{k}(1 - e^{-\frac{2k}{m}s})$$

$$\Longrightarrow v = \sqrt{y} = \sqrt{\frac{mg}{k}}\sqrt{1 - e^{-\frac{2k}{m}s}}$$

$$\Longrightarrow v_T = v(s_0) = \sqrt{\frac{mg}{k}}\sqrt{1 - e^{-\frac{2k}{m}s_0}}$$

Remark: The general solution of $y' = -\frac{2k}{m}y + 2g$ can also be obtained using the observation that $y \equiv mg/k$ is a particular (constant) solution.

b) When $m = 620\,\mathrm{g}$,

$$v_l = \lim_{s\to\infty}\sqrt{\frac{mg}{k}}\sqrt{1 - e^{-\frac{2k}{m}s}} = 20$$

$$\Longrightarrow \sqrt{\frac{mg}{k}} = 20$$

Assuming that $g = 10\,\mathrm{m/s^2}$, we have $\frac{2k}{m} = \frac{1}{20}$

$$\Longrightarrow v_T = 20\sqrt{1 - e^{-\frac{1}{20}s_0}}$$

(i)

$$v_T = 50\%\, v_l \Longrightarrow \sqrt{1 - e^{-\frac{1}{20}s_0}} = 50\% \Longrightarrow s_0 = -20\ln\frac{3}{4}$$

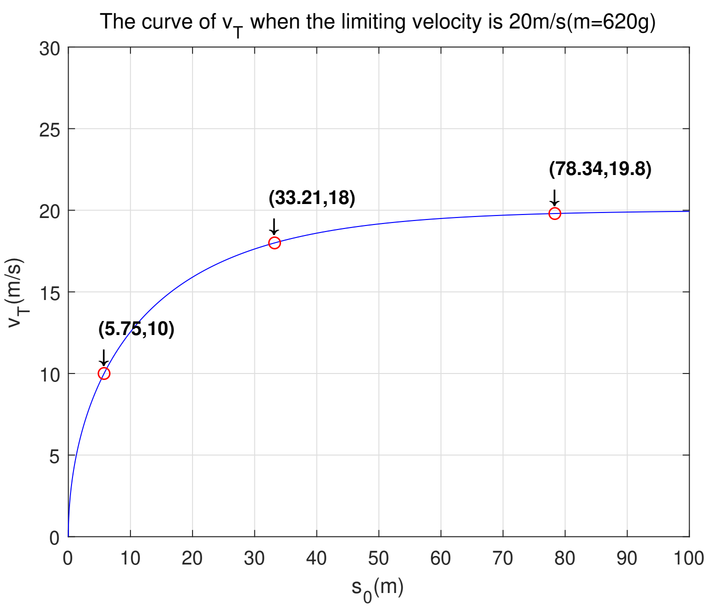
(ii)

$$v_T = 90\%\, v_l \Longrightarrow \sqrt{1 - e^{-\frac{1}{20}s_0}} = 90\% \Longrightarrow s_0 = -20\ln\frac{19}{100}$$

(iii)

$$v_T = 99\%\, v_l \Longrightarrow \sqrt{1 - e^{-\frac{1}{20}s_0}} = 99\% \Longrightarrow s_0 = -20\ln\frac{199}{10000}$$

The graph of $v_T(s_0) = 20\sqrt{1 - e^{-\frac{1}{20}s_0}}$ as a function of s_0 is shown below, with three points indicating the corresponding s_0 for which the basketball reaches 50%, 90%, and 99% of its limiting velocity.



Since we have used an approximation of g with only 1 significant digit, we cannot expect the values of s_0 to be more accurate. All we can say is that the basketball reaches 50%, 90%, 99% of its limiting velocity for heights of approximately 6 m, 30 m, 80 m, respectively.

12 a) Dividing $(\star)$ by $e^{\lambda_r t}$ and solving for c_r gives

$$c_r = -c_1 e^{(\lambda_1-\lambda_r)t} - c_2 e^{(\lambda_2-\lambda_r)t} - \cdots - c_{r-1} e^{(\lambda_{r-1}-\lambda_r)t}. \tag{1}$$

Since $\lambda_i - \lambda_r < 0$ for $1 \leq i \leq r-1$, we have $\lim_{t\to+\infty} e^{(\lambda_i-\lambda_r)t} = 0$ for $1 \leq i \leq r-1$. Hence the right-hand side of (1) tends to zero for $t \to +\infty$, while the left-hand side is a non-zero constant. This obvious contradiction proves that the functions f_λ, $\lambda \in \mathbb{R}$, are linearly independent over $\mathbb{R}$.

b) In the complex case $\lambda_i = \alpha_i + i\beta_i$ ($\alpha_i, \beta_i \in \mathbb{R}$) we have

$$e^{\lambda_i t} = e^{\alpha_i t + i(\beta_i t)}, \quad \left|e^{\lambda_i t}\right| = e^{\alpha_i t}.$$

Assuming that $\alpha_r > \alpha_i$ for $1 \leq i \leq r-1$ and $c_r \neq 0$, we can still divide $(\star)$ by $e^{\alpha_r t}$ and obtain a contradiction in a similar way. But, since different λ_i's may have the same real part, this is not sufficient for a proof of linear independence.

However, we can argue as follows: Differentiating $(\star)$ $r-1$-times gives the system of identities

$$\begin{array}{ccccccc} c_1 e^{\lambda_1 t} & + & \cdots & + & c_r e^{\lambda_r t} & = & 0, \\ c_1\lambda_1 e^{\lambda_1 t} & + & \cdots & + & c_r\lambda_r e^{\lambda_r t} & = & 0, \\ c_1\lambda_1^2 e^{\lambda_1 t} & + & \cdots & + & c_r\lambda_r^2 e^{\lambda_r t} & = & 0, \\ & & & & & \cdots & \\ c_1\lambda_1^{r-1} e^{\lambda_1 t} & + & \cdots & + & c_r\lambda_r^{r-1} e^{\lambda_r t} & = & 0. \end{array}$$

Setting $t = 0$ gives for $\mathbf{c} = (c_1, \dots, c_r)$ the linear system of equations $\mathbf{Ac} = \mathbf{0}$ with coefficient matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & \dots & 1 \\ \lambda_1 & \lambda_2 & \dots & \lambda_r \\ \lambda_1^2 & \lambda_2^2 & \dots & \lambda_r^2 \\ \vdots & \vdots & & \vdots \\ \lambda_1^{r-1} & \lambda_2^{r-1} & \dots & \lambda_r^{r-1} \end{pmatrix}.$$

Now $\mathbf{A}$ is a Vandermonde matrix and hence invertible; cf. any Linear Algebra book. (One can also compute the determinant of $\mathbf{A}$ recursively: Subtract Column 1 from the remaining columns and then expand along the first row. This leaves an $(n-1)\times(n-1)$ determinant, which has the factor $(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)\cdots(\lambda_r-\lambda_1)$ (since the columns have the factors $\lambda_j - \lambda_1$). After taking the factors out, the Vandermonde form (with 2nd row $(\lambda_2, \dots, \lambda_r)$) can be restored using suitable elementary row operations. By induction it then follows that $\det(\mathbf{A}) = \prod_{1\leq i<j\leq r}(\lambda_j - \lambda_i)$, which is obviously nonzero.)

It follows that $c_1 = c_2 = \cdots = c_r = 0$, i.e., the functions f_λ, $\lambda \in \mathbb{C}$, are linearly independent.

c) Let $\lambda_1, \dots, \lambda_r, \mu_1, \dots, \mu_t \in \mathbb{R}$ and $a_1, \dots, a_r, b_1, \dots, b_t \in \mathbb{R}$ be such that $0 \leq \lambda_1 < \cdots < \lambda_r$, $0 < \mu_1 < \cdots < \mu_t$, and

$$a_1 c_{\lambda_1} + \cdots + a_r c_{\lambda_r} + b_1 s_{\mu_1} + + \cdots + b_t s_{\mu_t} = 0 \quad \text{in} \quad \mathbb{R}^{\mathbb{R}}. \tag{2}$$

Since $\cos(\lambda x) = \frac{1}{2}(\mathrm{e}^{\mathrm{i}\lambda x} + \mathrm{e}^{-\mathrm{i}\lambda x})$, $\sin(\lambda x) = \frac{1}{2\mathrm{i}}(\mathrm{e}^{\mathrm{i}\lambda x} - \mathrm{e}^{-\mathrm{i}\lambda x})$, we have $c_\lambda = \frac{1}{2}(f_{\mathrm{i}\lambda} + f_{-\mathrm{i}\lambda})$, $s_\lambda = \frac{1}{2\mathrm{i}}(f_{\mathrm{i}\lambda} - f_{-\mathrm{i}\lambda})$. Inserting this into (2) gives a complex linear combination of the functions f_λ, which is equal to zero. By Part b), all the coefficients must be zero.

If $\lambda_1 = 0$ then, since $c_0(t) = 1 = f_0(t)$, the function f_0 appears in the complex linear combination with coefficient a_0, and hence $a_0 = 0$.

If λ_1 is not equal to any of the numbers $\mu_1, \dots, \mu_t$, then both $f_{\mathrm{i}\lambda_1}$, $f_{-\mathrm{i}\lambda_1}$ appear in the complex linear combination with coefficient $a_1/2$, showing that $a_1 = 0$.

Arguing similarly for $\lambda_2, \dots, \lambda_r, \mu_1, \dots, \mu_t$, we see that the only remaining case is $\lambda_i = \mu_j$ for some i, j. W.l.o.g. we may assume $\lambda_1 = \mu_1 = \lambda$. Then the coefficient of $f_{\pm\mathrm{i}\lambda}$ in the complex linear combination is clearly the same as in

$$a_1 c_{\lambda_1} + b_1 s_{\mu_1} = \frac{a_1}{2}\,(f_{\mathrm{i}\lambda} + f_{-\mathrm{i}\lambda}) + \frac{b_1}{2\mathrm{i}}\,(f_{\mathrm{i}\lambda} - f_{-\mathrm{i}\lambda}) = \frac{a_1 - \mathrm{i}b_1}{2}\,f_{\mathrm{i}\lambda} + \frac{a_1 + \mathrm{i}b_1}{2}\,f_{-\mathrm{i}\lambda}.$$

It follows that $\frac{a_1 - \mathrm{i}b_1}{2} = \frac{a_1 + \mathrm{i}b_1}{2} = 0$ and hence $a_1 = b_1 = 0$.

In all we have shown that (2) implies $a_1 = \dots = a_r = b_1 = \dots = b_t = 0$, i.e., the functions c_λ, $\lambda \geq 0$, and s_λ, $\lambda > 0$, are collectively linearly independent.

Differential Equations (Math 285)

H13 Determine the general solution of the following ODE's in terms of $y(0)$ (three answers suffice).

a) $\mathrm{d}y/\mathrm{d}t = e^{y+t}$;
b) $\mathrm{d}y/\mathrm{d}t = ty + y + t$;
c) $\mathrm{d}y/\mathrm{d}t = (\cos t)y + 4\cos t$;
d) $\mathrm{d}y/\mathrm{d}t = t^m y^n$ $(m, n \in \mathbb{Z})$.

H14 For the following ODE's, solve the corresponding IVP with $y(0) = 1$.

a) $\mathrm{d}y/\mathrm{d}t = -4ty$;
b) $\mathrm{d}y/\mathrm{d}t = t\,y^3$;
c) $(1 + t)\mathrm{d}y/\mathrm{d}t = 4y$.

H15 Determine all maximal solutions of $t^2\,y' = y^2$ and decide for which points $(t_0, y_0) \in \mathbb{R}^2$ the IVP $t^2\,y' = y^2 \wedge y(t_0) = y_0$ has no solution/exactly one solution/more than one solution.

H16

a) According to `worldometers.info`, the world's population on Ju1y 1, 2020 was about 7.79 billion, with a 1.05 % increase since July 1, 2019. Use this data to determine a new logistic model for the world's population growth, and compare with that of the lecture. What is the limiting population according to the new model?

b) Show that the graph of $y(t) = a/(d\,\mathrm{e}^{-at} + b)$ $(a, b, d > 0)$ is point-symmetric to its inflection point.

Hint: A superb way to solve this exercise is to observe that the mirror image of a solution curve w.r.t. its inflection point represents a solution as well and use the uniqueness of solutions of associated IVP's.

H17

a) Explain how to adapt the analysis of the harvesting equation in the lecture to $y' = ay^2 + by + c$ with $a, b, c \in \mathbb{R}$ and $a > 0$.

b) Sketch the solution curves of (i) $y' = y^2 - y + 1$, (ii) $y' = y^2 + 2y + 1$, (iii) $y' = y^2 + y - 2$ without actually computing solutions. Steady-state solutions and inflection points (if any) should be drawn exactly.

H18 The ODE $y' = a(t)y - b(t)y^n$, $n \in \mathbb{R} \setminus \{0, 1\}$ is called *Bernoulli's differential equation.*

a) Show that for an appropriate choice of $\beta \in \mathbb{R}$ the substitution $z = y^\beta$ turns Bernoulli's differential equation into a linear 1st-order ODE (which can be solved by the usual methods).

b) Solve the IVP $y' = 4y - y^3 \wedge y(0) = 1$ by the method suggested in a).

c) Investigate the asymptotic stability of the steady-state solutions of the ODE in b).

Due on Thu Mar 7, 10 am

Solutions

13 a)

$$e^{-y}\,\mathrm{d}y = e^{t}\,\mathrm{d}t$$

Integrating both sides of the equation, we get

$$\int_{y(0)}^{y} e^{-r}\,\mathrm{d}r = \int_{0}^{t} e^{s}\,\mathrm{d}s$$
$$-e^{-y} + e^{-y(0)} = e^{t} - 1$$

Finally, we obtain

$$y(t) = -\ln(e^{-y(0)} + 1 - e^{t}), \quad t < \ln(1 - e^{-y(0)}).$$

Remark: When determining the solution, one can also use indefinite integration $\int e^{-y}\,\mathrm{d}y = e^{t}\,\mathrm{d}t + C$ and determine C in terms of $y(0)$. This applies to the subsequent exercises as well.

b) Rewrite the ODE in the form of $y' = a(t)y + b(t)$:

$$y' = (t+1)y + t$$

According to the particular solution formula,

$$\begin{aligned}
y_p(t) &= e^{\frac{t^2}{2}+t} \int_0^t s\, e^{-(\frac{s^2}{2}+s)}\,\mathrm{d}s \\
&= e^{\frac{t^2}{2}+t} \left(\int_0^t (s+1)e^{-(\frac{s^2}{2}+s)}\,\mathrm{d}s - \int_0^t e^{-(\frac{s^2}{2}+s)}\,\mathrm{d}s \right) \\
&= -e^{\frac{t^2}{2}+t} \left(e^{-(\frac{t^2}{2}+t)} - 1 \right) - e^{\frac{1}{2}} \int_0^t e^{-\left(\frac{s^2}{2}+s+\frac{1}{2}\right)}\,\mathrm{d}s \\
&= e^{\frac{t^2}{2}+t} - 1 - e^{\frac{1}{2}} \int_0^t e^{-\left(\frac{s+1}{\sqrt{2}}\right)^2}\,\mathrm{d}s,
\end{aligned}$$

and the “homogeneous solution” is

$$y_h(t) = y(0)\, e^{\frac{t^2}{2}+t}$$

Since $y_p(0) = 0$, the general solution in terms of $y(0)$ is

$$y(t) = y(0)e^{\frac{t^2}{2}+t} - 1 - e^{\frac{1}{2}} \int_0^t e^{-\left(\frac{s+1}{\sqrt{2}}\right)^2}\,\mathrm{d}s\,.$$

Remark: It is not necessary to rewrite the integrand occuring in $y_p(t)$ in the particular form shown above, but at least this shows the relation with the incomplete Gauss integral (or the so-called error function). The simple answer is $y(t) = y_p(t) + y_h(t)$, $t \in \mathbb{R}$, with y_p, y_h as above.

c) According to the particular solution formula,

$$
\begin{aligned}
y_p(t) &= e^{\sin(t)} \int_0^t 4\cos(s) e^{-\sin(s)}\ \mathrm{d}s && (1)\\
&= -4e^{\sin(t)}(e^{-\sin(t)} - 1) && (2)\\
&= 4e^{\sin(t)} - 4, && (3)
\end{aligned}
$$

and the "homogeneous solution" is

$$y_h(t) = y(0)e^{\sin(t)}.$$

The general solution is then

$$y(t) = (y(0) + 4)e^{\sin(t)} - 4.$$

The general form $y(t) = C\mathrm{e}^{\sin t} - 4$, $C \in \mathbb{R}$, also follows from the observation that $y(t) \equiv -4$ is a particular solution.

d) There is the constant solution $y = 0$, and for $y \neq 0$ we can separate:

$$\frac{dy}{y^n} = t^m \,\mathrm{d}t\,.$$

Integrating both sides, we get

$$
\begin{aligned}
\int_{y(0)}^{y} \frac{1}{r^n}\,\mathrm{d}r &= \int_0^t s^m\,\mathrm{d}s\\
-\frac{1}{(n-1)y^{n-1}} + \frac{1}{(n-1)y(0)^{n-1}} &= \frac{t^{m+1}}{m+1}, \quad (n \neq 1, \quad m \neq -1).
\end{aligned}
$$

Then, we obtain the general solution

$$y(t) = \left[(n-1)\left(\frac{1}{(n-1)y(0)^{n-1}} - \frac{t^{m+1}}{m+1}\right)\right]^{-\frac{1}{n-1}} \quad (n \neq 1, \quad m \neq -1).$$

Next, we deal with the special cases:

i) $n = 1, \quad m = -1$

$$
\begin{aligned}
\frac{\mathrm{d}y}{y} &= \frac{\mathrm{d}t}{t}\\
\ln|y| &= \ln|t| + C
\end{aligned}
$$

Finally, we obtain, with a different parameter $C' \in \mathbb{R}$,

$$y(t) = C't, \quad t \in (-\infty, 0) \text{ or } t \in (0, +\infty).$$

$y(0)$ is not defined in this case.

ii) $n = 1, \quad m \neq -1$

$$\frac{\mathrm{d}y}{y} = t^m dt$$

Integrating both sides, we get

$$\int_{y(0)}^{y} \frac{1}{r}\,\mathrm{d}r = \int_0^t s^m\,\mathrm{d}s,$$

$$\ln|y| - \ln|y(0)| = \frac{t^{m+1}}{m+1}.$$

Finally, noting that $y(t)$ and $y(0)$ must have the same sign, we obtain

$$y(t) = y(0)e^{\frac{t^{m+1}}{m+1}}, \quad t \in \mathbb{R}.$$

.

iii) $n \neq 1, \quad m = -1$

$$\frac{\mathrm{d}y}{y^n} = \frac{\mathrm{d}t}{t}$$

Integrate both sides, we get

$$-\frac{1}{(n-1)y^{n-1}} = \ln|t| + C$$

Finally, we obtain

$$y(t) = (-(n-1)(\ln|t| + C))^{-\frac{1}{n-1}}, \quad t < -e^{-C} \text{ or } t > e^{-C}.$$

$y(0)$ is not defined in this case.

14 a) $dy/dt = -4ty$

This is a homogeneous linear ODE, so we get

$$y(t) = Ce^{-2t^2}$$

Plugging into the IVP $y(0) = 1$, we can obtain the solution as

$$y(t) = e^{-2t^2}, \quad t \in \mathbb{R}.$$

b) $dy/dt = ty^3$

This is a separable ODE, so we can write

$$\frac{dy}{y^3} = tdt$$

$$\int_1^y \frac{1}{r^3}\,dr = \int_0^t s\,ds$$

The solution is

$$y(t) = (1-t^2)^{-\frac{1}{2}}, \quad -1 < t < 1.$$

c) $(1+t)dy/dt = 4y$
Rewrite the ODE as

$$y' = \frac{4}{t+1}y.$$

We use the "homogeneous solution formula" to get

$$y(t) = Ce^{4\ln|t+1|} = C(t+1)^4.$$

Plugging into the IVP $y(0)=1$, we obtain the solution as

$$y(t) = (t+1)^4, \quad t \in \mathbb{R}.$$

15 We can rewrite this separable ODE as

$$\frac{dy}{y^2} = \frac{dt}{t^2} \quad (y, t \neq 0)$$

Integrating both sides of the above equation, we get

$$\int_{y_0}^{y} \frac{1}{\eta^2}\,\mathrm{d}\eta = \int_{t_0}^{t} \frac{1}{\tau^2}\,\mathrm{d}\tau$$
$$-\frac{1}{y} + \frac{1}{y_0} = -\frac{1}{t} + \frac{1}{t_0}$$

which gives

$$y(t) = \frac{t_0 y_0}{\frac{t_0 y_0}{t} - (y_0 - t_0)} = \frac{(t_0 y_0)t}{t_0 y_0 - (y_0 - t_0)t} = \frac{t}{1 - Ct} \quad \text{with} \quad C := \frac{y_0 - t_0}{t_0 y_0}.$$

Removal of the coordinate axes splits the (t, y)-plane into 4 quadrants ("small rectangles" in the parlance of the lecture), and every solution that is contained entirely in one of these quadrants must be of this form.
For $t = 0$ we must have $y(t) = 0$ (from $t^2 y' = y^2$).
There is the constant solution $y(t) = 0$, $t \in \mathbb{R}$.

1) $t_0 = y_0 \neq 0$

$$y(t) = \begin{cases} t, & t \neq 0 \\ 0, & t = 0 \end{cases}$$

And $y(0) = 0$ fits with the expression $y(t) = t$, so we can write the solution as $y(t) = t$, $t \in \mathbb{R}$.
There is only one solution.

2) $t_0 = y_0 = 0$
Every solution of the ODE defined at $t = 0$ must satisfy $y(0) = 0$ (see above). Non-constant (maximal) solutions are

$$y(t) = \frac{t}{1 - Ct}, \quad C \in \mathbb{R}.$$

with domain $\mathbb{R}$ if $C = 0$, $(-\infty, 1/C)$ if $C > 0$, and $(1/C, +\infty)$ if $C < 0$. In particular there are an infinite number of solutions.

3) $t_0 = 0, y_0 \neq 0$
   This IVP contradicts $y(0) = 0$. Therefore, there is no solution.

4) $t_0 \neq 0, y_0 = 0$
   The solution is

$$y(t) = 0$$

   Therefore, there is only one solution.

5) $0 \neq t_0 \neq y_0 \neq 0$

$$y(t) = \frac{t}{1 - \frac{y_0 - t_0}{t_0 y_0} t}, \quad t \neq \frac{t_0 y_0}{y_0 - t_0}$$

   Therefore, there is only one solution.

In conclusion, the IVP $t^2 y' = y^2 \wedge y(t_0) = y_0$ has

1) no solution when $t_0 = 0,\ y_0 \neq 0$;

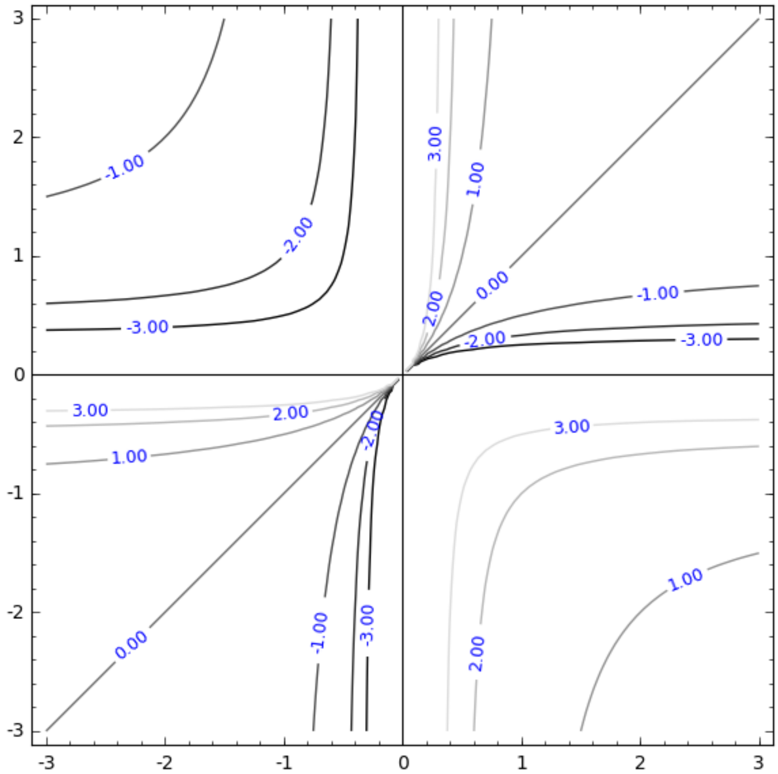
2) infinitely many (maximal) solutions when $t_0 = y_0 = 0$;

3) exactly one (maximal) solution otherwise.

Particular maximal solutions are

$$\begin{aligned} y(t) &= 0, && t \in \mathbb{R}; \\ y(t) &= t, && t \in \mathbb{R}; \\ y_C^-(t) &= \frac{t}{1 - Ct}, && t \in (-\infty, 1/C); \\ y_C^+(t) &= \frac{t}{1 - Ct}, && t \in (1/C, +\infty). \end{aligned}$$

The 3rd and 4th type of solutions exist for any real number $C \neq 0$.
A better picture can be obtained by solving $y = t/(1 - Ct)$ for C, which gives $C = (y - t)/(ty)$ and shows that $F(t, y) = (y - t)/(ty)$ provides a first integral for the given ODE.



From the contour plot of F you can see that all non-constant solutions that are defined at $t = 0$ share the same tangent at $(0, 0)$. This also follows from $\frac{\mathrm{d}}{\mathrm{d}t} \frac{t}{1-Ct} = \frac{1}{(1-Ct)^2}$. Solutions $y_C^+(t)$ with $C > 0$ fill the 4th quadrant, solutions $y_C^-(t)$ with $C > 0$ fill the region above $y = t$ in the 1st and 3rd quadrant, etc. All these properties can also be derived with some effort from the formulas.
For determining all maximal solutions we must also consider the possibility of glueing together branches of solutions at the origin. Since the tangent of a non-constant solution at the origin must be $y = t$, any solution in the 3rd quadrant "ending at $(0, 0)$" can be glued together with any solution in the 1st quadrant "starting at $(0, 0)$" to form a maximal solution. This gives a two-parameter family of maximal solutions (see the picture), which may also have a finite domain or domain $\mathbb{R}$.
Remark: With the Existence and Uniqueness Theorems now at hand, we can easily get a complete qualitive picture. Rewriting $t^2 y' = y^2$ as $y^2 \,\mathrm{d}t - t^2 \,\mathrm{d}y = 0$, we see that the origin $(t_0, y_0) = (0, 0)$ is the only singular point, and hence that through any other point (t_0, y_0) there passes precisely one integral curve (solution curve) locally.[1] For points $(0, y_0)$ with $y_0 \neq 0$ this is the curve $t = 0$, which cannot be seen from $t^2 y' = y^2$, because it can be parametrized only as $t(y)$.

16 a) The new logistic model is $y(t) = \frac{a}{d' \mathrm{e}^{-a(t-2020)} + b}$ with $a = 0.029$ (natural reproduction rate of humans) and b, d' determined from $\frac{y'(2020)}{y(2020)} = a - b\, y(2020) = 0.0105$, $y(2020) =$

[1]By this we mean that there exists a neighborhood U of (t_0, y_0) such that the modified ODE with domain U has uniqueness of integral curves through (t_0, y_0). The set U must be chosen in such a way that it doesn't contain the origin.

$\frac{a}{d'+b} = 7.79 \times 10^9$. The solution is $b = \frac{37}{1558} \times 10^{-10}$, $d' = \frac{21}{1558} \times 10^{-10}$, so that

$$y(t) = \frac{0.029 \times 10^{10}}{\frac{21}{1558}\,\mathrm{e}^{-0.029(t-2020)} + \frac{37}{1558}}$$

and the limiting population is $a/b \approx 12.2$ billion people.

b) With the Hint, we want to prove that the mirror image of a solution curve w.r.t. its inflection point represents a solution as well and use the uniqueness of solutions of associated IVP's.
The function of the mirror image is

$$g(t) = \frac{a}{b} - \frac{a}{de^{-a(2t_h-t)} + b}$$

where $t_h = (\ln d - \ln b)/a$.
First, we prove that $g(t)$ is a solution to the ODE $y' = ay - by^2$.

$$g'(t) = -\frac{a^2 de^{a(2t_h-t)}}{(de^{a(2t_h-t)} + b)^2}$$

and

$$\begin{aligned} ag(t) - bg^2(t) &= \frac{a^2}{b} - \frac{a^2}{de^{-a(2t_h-t)} + b} - b(\frac{a^2}{b^2} - \frac{2a^2}{b(de^{-a(2t_h-t)} + b)} + \frac{a^2}{(de^{-a(2t_h-t)} + b)^2}) \\ &= \frac{-a^2 de^{-a(2t_h-t)} - a^2 b + 2a^2 b - a^2 b}{(de^{-a(2t_h-t)} + b)^2} \\ &= -\frac{a^2 de^{-a(2t_h-t)}}{(de^{-a(2t_h-t)} + b)^2} \end{aligned}$$

Thus, $g'(t) = ag(t) - bg^2(t)$, which means $g(t)$ is also a solution to the ODE $y' = ay - by^2$.
Then, we will use the uniqueness of the solution of th IVP $y' = ay - by^2 \wedge y(t_h) = a/2b$. Since the original solution curve has the inflection point $(t_h, a/2b)$, it shares the same IVP with the mirror image $g(t)$. The logistic equation has a unique solution for any given IVP, so $y(t) = g(t)$.

Remark: The computation can be simplified a little by using the observation that $y(t)$ solves $y' = ay - by^2$ iff $t \mapsto y(t - t_0)$, $t_0 \in \mathbb{R}$, does.

17 a) It should be noted that the analysis in the lecture used the notation $y' = ay - by^2 - h$, where $a, b, h > 0$. However, the parabola $f(y) = y' = ay^2 + by + c$, $a > 0$, is a vertically flipped version of that considered in the lecture. This discrepancy will lead to different behaviors of the solution curves.
The discriminant is $\Delta = b^2 - 4ac$. For $\Delta \geq 0$, there are the steady-state solutions

$$y_1 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

$$y_2 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

where $0 < y_1 \leq y_2$.

i) $c < b^2/4a$
If the initial condition $y(t_0)$ satisfies $y_1 < y(t_0) < y_2$, then $y(t)$ decreases and $\lim_{t\to\infty} y(t) = y_1$.
If $y(t_0) > y_2$, then $y(t)$ increases to ∞. If $y(t_0) < y_1$, then $y(t)$ increases and $\lim_{t\to\infty} y(t) = y_1$.

ii) $c = b^2/4a$
If $y(t_0) > -b/2a$, then $y(t)$ increases to ∞. If $y(t_0) < -b/2a$, then $y(t)$ increases and $\lim_{t\to\infty} y(t) = y_1$.

iii) $c > b^2/4a$
Regardless of the initial condition, $y(t)$ will increase to ∞.

b) i) $y' = y^2 - y + 1$

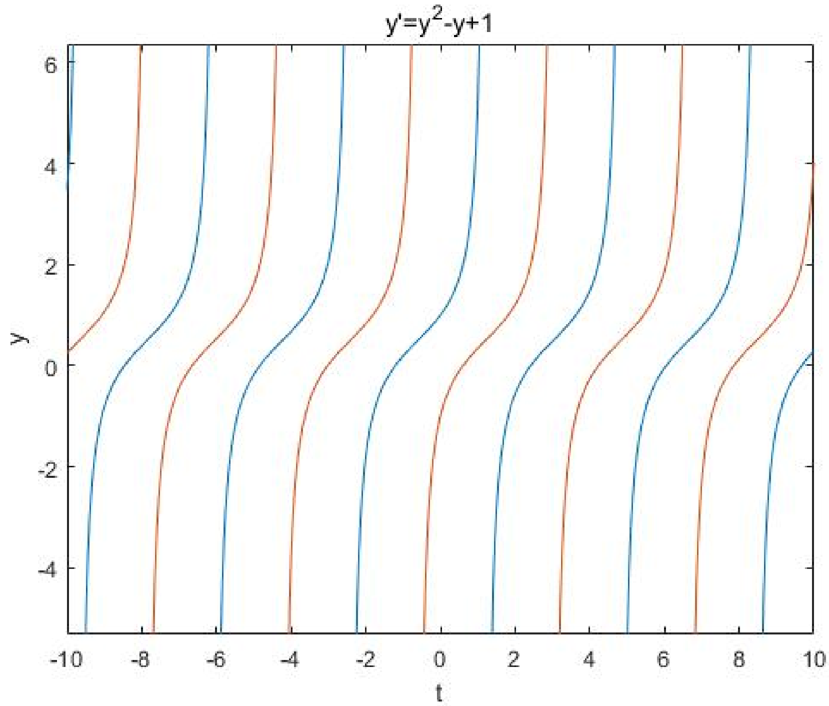


Figure 1: $y' = y^2 - y + 1$

ii) $y' = y^2 + 2y + 1$

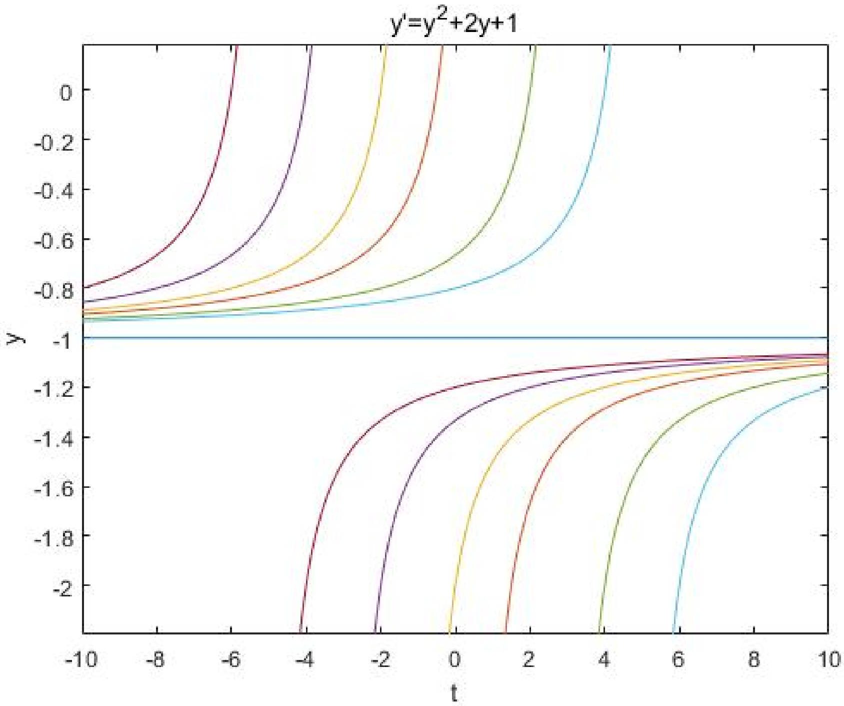


Figure 2: $y' = y^2 + 2y + 1$

iii) $y' = y^2 + y - 2$

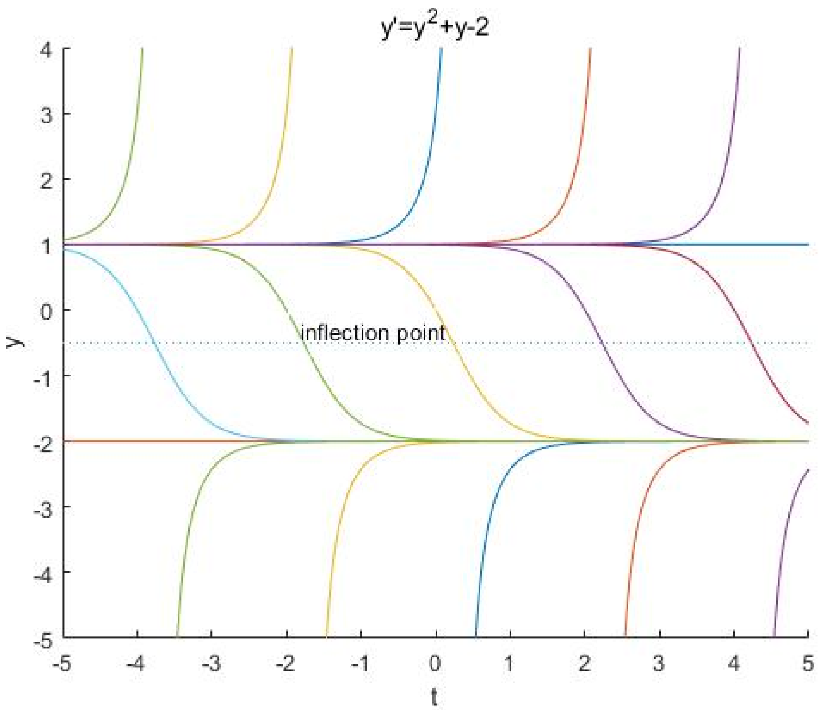


Figure 3: $y' = y^2 + y - 2$

18 a) We can write

$$\frac{dz}{dt} = \beta y^{\beta-1}\frac{dy}{dt}.$$

Then, we get

$$\frac{dy}{dt} = \frac{1}{\beta} y^{1-\beta} \frac{dz}{dt}.$$

Substituting the above expression into the ODE, we get

$$\begin{aligned} \frac{1}{\beta} y^{1-\beta} \frac{dz}{dt} &= a(t)y - b(t)y^n, \\ z' &= \beta a(t) y^\beta - \beta b(t) y^{n+\beta-1}, \\ z' &= \beta a(t) z - \beta b(t) y^{n+\beta-1}. \end{aligned}$$

Then, setting $\beta = 1 - n$, we can obtain the 1st-order linear ODE

$$z' = \beta a(t) z - \beta b(t)$$

for $z(t) = y(t)^{1-n}$. Depending on n, the 1-1 correspondence between solutions of both ODEs may only hold for a smaller domain, e.g., for general $n > 1$ we need to restrict to $y > 0$ (except for certain integers n).

b) Setting $\beta = 1 - 3 = -2$, we can rewrite the ODE as

$$z' = -8z + 2.$$

The corresponding IVP is $z(0) = y(0)^{-2} = 1$.
Then, we can get its solution as

$$z(t) = \frac{3}{4} e^{-8t} + \frac{1}{4}.$$

Since $z = y^\beta = y^{-2}$,

$$y(t) = \pm z(t)^{-\frac{1}{2}} = \pm \left(\frac{3}{4} e^{-8t} + \frac{1}{4} \right)^{-\frac{1}{2}}$$

Because $y(0) = 1$, we eliminate the negative solution, leaving

$$y(t) = \left(\frac{3}{4} e^{-8t} + \frac{1}{4} \right)^{-\frac{1}{2}} = \frac{2}{\sqrt{1 + 3\,\mathrm{e}^{-8t}}}, \quad t \in \mathbb{R}.$$

c) The steady-state solution is $z(t) = 1/4$, corresponding to $y(t) = \pm 2$.
The general solution to the ODE in b) is $y(t) \equiv 0$ and the non-constant solutions

$$y_1(t) = -\left[\left(y^{-2}(0) - \frac{1}{4} \right) e^{-8t} + \frac{1}{4} \right]^{-\frac{1}{2}},$$

and

$$y_2(t) = \left[\left(y^{-2}(0) - \frac{1}{4} \right) e^{-8t} + \frac{1}{4} \right]^{-\frac{1}{2}}.$$

$\lim_{t\to\infty} y_1(t) = -2$, and $\lim_{t\to\infty} y_1(t) = 2$.

If the initial condition is $y(0) = y_0 < 0$, then the solution will be $y_1(t)$, so $\lim_{t\to\infty} y(t) = -2$;
if the initial condition is $y(0) = y_0 > 0$, then the solution will be $y_2(t)$, so $\lim_{t\to\infty} y(t) = 2$.
This shows that both $y = -2$ and $y = 2$ are asymptotically stable.
The third steady state solution $y(t) \equiv 0$ is unstable. This follows from the cases $y(0) > 0$ and $y(0) < 0$ covered above.

Differential Equations (Math 285)

H19 Find integrating factors for the following ODE's and determine their integral curves.

a) $e^x(x+1)\,dx + (y\,e^y - x\,e^x)\,dy = 0$;

b) $y(y+2x+1)\,dx - x(2y+x-1)\,dy = 0$.

H20 An ODE $M(x,y)\,dx + N(x,y)\,dy = 0$ is said to be *homogeneous* if M and N are homogeneous functions of the same degree, i.e., there exists $d \in \mathbb{R}$ such that $M(\lambda x, \lambda y) = \lambda^d M(x,y)$ and $N(\lambda x, \lambda y) = \lambda^d N(x,y)$ for all x, y, and λ.

a) Show that the substitution $z = y/x$ (or $z = x/y$) transforms any homogeneous ODE into a separable ODE.

b) Solve the following ODE's in implicit form (answering two of (i)–(iii) suffices):

(i) $(x+y)\,dx - (x+2y)\,dy = 0$; (ii) $(x-2y)\,dx + y\,dy = 0$;
(iii) $(x^2+y^2)\,dx + 3xy\,dy = 0$; (iv) $(x-y-1)\,dx + (x+4y-6)\,dy = 0$.

H21 a) Determine the orthogonal trajectories of the family of circles through $(1,0)$ and $(-1,0)$.

Hint: The midpoint of such a circle is on the y-axis, and it is best to use its y-coordinate as parameter C.~~Use the y-coordinate of the intersection point with the y-axis as parameter C, i.e., $(0,C)$ is on the circle.~~

b) Determine the orthogonal trajectories of the family of parabolas $y = k\,x^2 - \frac{1}{4k}$, $k > 0$ (confocal parabolas with focus in $(0,0)$, the y-axis as axis, and open on the top). What is the relation with Exercise W24 d) on Worksheet 8 of Calculus III in Fall 2023?

H22 Analyze the alternative model $dy/dt = ay - by^2 - Ey$ $(a,b,E > 0)$ for harvesting a population (individuals are removed at a rate proportional to the current size of the population). Which rates E are sustainable? How to choose E in order to maximize the *yield* Ey in the long run?

H23 *Optional exercise*

The task of this exercise is to show the Cauchy-Hadamard formula

$$R = \frac{1}{L}, \quad L = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$$

(with the conventions $1/0 = \infty$, $1/\infty = 0$) for the radius of convergence R of a (complex) power series $\sum_{n=0}^{\infty} a_n(z-a)^n$. Here $L = \limsup_{n\to\infty} x_n \in [-\infty, +\infty]$ (*limit superior*) denotes the largest accumulation point of a real sequence (x_n), i.e., for every $\epsilon > 0$ there are only finitely many indexes n satisfying $x_n \geq L + \epsilon$ but no real number $L' < L$ has this property (with suitable modifications for $L = \pm\infty$).

PLEASE TURN OVER

a) If $L = \infty$ (i.e., $\sqrt[n]{|a_n|}$ is unbounded), show that $\sum_{n=0}^{\infty} a_n(z-a)^n$ converges only for $z = a$.

b) If $L = 0$ (i.e., $\sqrt[n]{|a_n|}$ converges to zero), show that $\sum_{n=0}^{\infty} a_n(z-a)^n$ converges for all $z \in \mathbb{C}$.

c) If $0 < L < \infty$, show that $\sum_{n=0}^{\infty} a_n(z-a)^n$ converges for $|z-a| < 1/L$ and diverges for $|z-a| > 1/L$.

H24 *Optional exercise*

For $s \in \mathbb{C}$ consider the *binomial series*

$$\mathrm{B}_s(z) = \sum_{n=0}^{\infty} \binom{s}{n} z^n = \sum_{n=0}^{\infty} \frac{s(s-1)\cdots(s-n+1)}{1\cdot 2\cdots n}\, z^n.$$

a) Show that for $s \notin \{0,1,2,\dots\}$ the binomial series has radius of convergence $R = 1$.

b) Show that $\mathrm{B}_s(x) = (1+x)^s$ for $s \in \mathbb{C}$ and $-1 < x < 1$.
Hint: $x \mapsto (1+x)^s = \mathrm{e}^{s\ln(1+x)}$ is a solution of the IVP $y' = \frac{s}{1+x}\, y$, $y(0) = 1$. Show that the same is true of $x \mapsto \mathrm{B}_s(x)$; cf. also [Ste16], Ch. 11.10, Ex. 85.

c) Show $\mathrm{B}_s(z) = (1+z)^s$ for $s, z \in \mathbb{C}$ with $|z| < 1$.
Hint: Probably the easiest way to solve this part is to use the same idea as in b): Show that $z \mapsto \mathrm{B}_s(z)$ and $z \mapsto (1+z)^s = \mathrm{e}^{s\log(1+z)}$ both satisfy $y' = \frac{s}{1+z}\, y$ for $|z| < 1$ and $y(0) = 1$, and that the solution of this complex IVP is unique. Since we haven't discussed complex differentiation and ODE's in any depth, it is important that you justify carefully every step of your solution.

Due on Thu Mar 14, 10 am

The optional exercises can be handed in until Thu Mar 21, 10 am.

Solutions

19 a) According to the equation,

$$M = \mathrm{e}^x(x+1), \quad N = ye^y - xe^x,$$
$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}.$$

This is not an exact ODE and an appropriate integrating factor is needed. Computing the following equation, we find that

$$\frac{1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{(x+1)e^x}{(x+1)e^x} = 1 = g(y)$$

Thus, there exists a suitable integrating factor $\mu(y)$ that is a function of y only, and μ satisfies the differential equation

$$\mu'(y) = -\mu(y)g(y) = -\mu(y).$$

Hence,

$$\mu(y) = e^{-y}$$

is a suitable integrating factor.

Multiplying the original equation by this integrating factor, we obtain

$$e^{x-y}(x+1)\,\mathrm{d}x + (y - xe^{x-y})\,\mathrm{d}y = 0,$$
$$M' = e^{x-y}(x+1), N' = y - xe^{x-y},$$
$$\frac{\partial M'}{\partial y} = -xe^{x-y} = \frac{\partial N'}{\partial x}.$$

This is an exact ODE.
Therefore, there exists a function φ so that

$$\frac{\partial \varphi}{\partial x} = M' = e^{x-y}(x+1), \quad \frac{\partial \varphi}{\partial y} = N' = y - xe^{x-y}.$$

Integrating the first equation with respect to x, we obtain that

$$\varphi(x,y) = \int e^{x-y}(x+1)dx + h(y) = xe^{x-y} + h(y).$$

Substituting $\varphi(x,y)$ into the second equation, we find that

$$\frac{\partial \varphi}{\partial y} = -xe^{x-y} + h'(y) = y - xe^{x-y} = N',$$

so $h'(y) = y$ and $h(y) = \frac{1}{2}y^2$. Thus the solution is given implicitly by

$$F(x,y) = xe^{x-y} + \frac{1}{2}y^2 = C, \quad C \in \mathbb{R}.$$

b) According to the equation,

$$M = y(y + 2x + 1), \quad N = -x(2y + x + 1),$$
$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}.$$

This is not an exact ODE and an appropriate integrating factor is needed. Computing the following expression,

$$\frac{1}{Ny - Mx}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{-3xy(x+y)}[(2y + 2x + 1) - (-2y - 2x + 1)] = \frac{4}{-3xy},$$

which depends only on xy, we can conclude that there exists a suitable integrating factor $\mu(xy)$ that is a function of xy only, and (subsitute $s = xy$) μ satisfies the differential equation

$$\mu'(s) = \mu(s)g(s) = \frac{4}{-3s}\,\mu(s).$$

Hence,

$$\mu(s) = e^{\int -\frac{4}{3s}\,\mathrm{d}s} = s^{-\frac{4}{3}},$$

and the integrating factor is $\mu(xy) = (xy)^{-\frac{4}{3}}$. Multiplying the original equation by this integrating factor, we obtain

$$(xy)^{-\frac{4}{3}}y(y + 2x + 1)\,\mathrm{d}x - (xy)^{-\frac{4}{3}}x(2y + x - 1)\,\mathrm{d}y = 0$$
$$M' = (xy)^{-\frac{4}{3}}y(y + 2x + 1), N' = -(xy)^{-\frac{4}{3}}x(2y + x - 1)$$
$$\frac{\partial M'}{\partial y} = \frac{\partial N'}{\partial x}$$

This is an exact ODE.
Therefore, there exists a function φ so that

$$\frac{\partial \varphi}{\partial x} = M' = (xy)^{-\frac{4}{3}}y(y + 2x + 1),$$
$$\frac{\partial \varphi}{\partial y} = N' = -(xy)^{-\frac{4}{3}}x(2y + x - 1).$$

Integrating the first equation with respect to x, we obtain

$$\varphi(x, y) = \int (xy)^{-\frac{4}{3}}y(y + 2x + 1)dx + h(y) = (y^{-\frac{1}{3}})[-3(y + 1)x^{-\frac{1}{3}} + 3x^{\frac{2}{3}}] + h(y).$$

Substituting $\varphi(x, y)$ into the second equation, we find that

$$\frac{\partial \varphi}{\partial y} = -3x^{-\frac{1}{3}}y^{-\frac{1}{3}} + (y + 1)x^{-\frac{1}{3}}y^{-\frac{4}{3}} - x^{\frac{2}{3}}y^{-\frac{4}{3}} + h'(y) = N',$$

so $h'(y) = 0$ and $h(y) = C$. Thus the solution is given implicitly by

$$F(x, y) = -3x^{-\frac{1}{3}}y^{-\frac{1}{3}} - 3x^{-\frac{1}{3}}y^{\frac{2}{3}} + 3x^{\frac{2}{3}}y^{-\frac{1}{3}} = C, \quad C \in \mathbb{R}.$$

20 a) The homogeneous ODE can be represented as

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0.$$

Substituting $z = y/x$ and using the defining property of homogeneous ODE's, we obtain

$$\begin{aligned}
& M(x,xz)\,\mathrm{d}x + N(x,xz)(x\,\mathrm{d}z + z\,\mathrm{d}x) = 0 \\
\iff\quad & x^d M(1,z)\,\mathrm{d}x + x^d N(1,z)(x\,\mathrm{d}z + z\,\mathrm{d}x) = 0 \\
\iff\quad & [M(1,z) + zN(1,z)]\,\mathrm{d}x + N(1,z)x\,\mathrm{d}z = 0 \\
\iff\quad & \frac{M(1,z) + zN(1,z)}{N(1,z)}\,\mathrm{d}x = -x\,\mathrm{d}z\,.
\end{aligned}$$

Hence, the homogeneous ODE can be converted to the separable ODE

$$\frac{\mathrm{d}z}{\mathrm{d}x} = -\frac{M(1,z) + zN(1,z)}{x\,N(1,z)}.$$

Alternative solution: We use the explicit form

$$y' = -\frac{M(x,y)}{N(x,y)} = -\frac{x^d\,M(1,y/x)}{x^d\,N(1,y/x)} = -\frac{M(1,y/x)}{N(1,y/x)} = f(y/x), \quad \text{say.}$$

The substitution $z = y/x$ gives

$$z' = \frac{y'x - y}{x^2} = \frac{f(z) - z}{x},$$

which is separable. (Inserting $f(z) = -M(1,z)/N(1,z)$ transforms this ODE into the ODE obtained above.)

Remark: The substitution $z = y/x$ requires $x \neq 0$. Some integral curves may be lost in this way, e.g., consider $x\,\mathrm{d}y - y\,\mathrm{d}x = 0$, which has the y-axis as an integral curve, but the transformed ODE $z' = 0$ doesn't reflect this.

b) (i) Substituting $z = y/x$ gives

$$(x+y)\,\mathrm{d}x - (x+2y)\,\mathrm{d}y = (x+xz)\,\mathrm{d}x - (x+2xz)(z\,\mathrm{d}x + x\,\mathrm{d}z) = 0$$

$$\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{1-2z^2}{(1+2z)x}$$

This is a separable ODE and can be solved in the usual way: There are the constant solutions $z = \pm\frac{1}{2}\sqrt{2}$, corresponding to $y = \pm\frac{1}{2}\sqrt{2}\,x$. Otherwise we get

$$\int \frac{1+2z}{1-2z^2}\,\mathrm{d}z = \int \frac{1}{x}\,\mathrm{d}x$$

$$\ln|x| = \frac{1}{2}\int \frac{1}{1-\sqrt{2}\,z}\,\mathrm{d}z + \frac{1}{2}\int \frac{1}{1+\sqrt{2}\,z}\,\mathrm{d}z + \int \frac{2z}{1-2z^2}\,\mathrm{d}z$$

$$\begin{aligned}
\ln|x| &= -\frac{1}{2\sqrt{2}}\ln\left|1-\sqrt{2}\,z\right| + \frac{1}{2\sqrt{2}}\ln\left|1+\sqrt{2}\,z\right| - \frac{1}{2}\ln\left|1-2z^2\right| + C \\
&= \ln\frac{\left|1+\sqrt{2}\,z\right|^{\frac{1}{2\sqrt{2}}-\frac{1}{2}}}{\left|1-\sqrt{2}\,z\right|^{\frac{1}{2\sqrt{2}}+\frac{1}{2}}} + C
\end{aligned}$$

$$|x| = \mathrm{e}^C \frac{\left|1+\sqrt{2}\,z\right|^{\frac{1}{4}\sqrt{2}-\frac{1}{2}}}{\left|1-\sqrt{2}\,z\right|^{\frac{1}{4}\sqrt{2}+\frac{1}{2}}} = \mathrm{e}^C \frac{|x|\left|x+\sqrt{2}\,y\right|^{\frac{1}{4}\sqrt{2}-\frac{1}{2}}}{\left|x-\sqrt{2}\,y\right|^{\frac{1}{4}\sqrt{2}+\frac{1}{2}}}$$

Dividing by $|x|$ and raising the equation to the 4th power, which only changes the constant, gives

$$\left|x - \sqrt{2}\,y\right|^{2+\sqrt{2}} = C' \left|x + \sqrt{2}\,y\right|^{2-\sqrt{2}}, \quad C' > 0.$$

This is probably the best form of the solution curves we can get. It includes the two special solutions $y = \pm\frac{1}{2}\sqrt{2}\,x$ as boundary cases $C' = 0$ and $C' = \infty$.

(ii) Substituting $z = y/x$ into the equation gives

$$\begin{aligned}(x - 2y)\,\mathrm{d}x + y\,\mathrm{d}y = (x - 2xz)\,\mathrm{d}x + xz(z\,\mathrm{d}x + x\,\mathrm{d}z) = 0,\\ \frac{\mathrm{d}z}{\mathrm{d}x} = \frac{-z^2 + 2z - 1}{zx} = -\frac{(z-1)^2}{zx}.\end{aligned}$$

This is a separable ODE, hence for $z \neq 1 (z = 1$ gives the solution $y = x)$ we can continue as usual:

$$\int \frac{-z}{1 - 2z + z^2}\,\mathrm{d}z = \int \frac{1}{x}\,\mathrm{d}x$$

$$\ln|x| = \int \frac{1 - z - 1}{1 - 2z + z^2}\,\mathrm{d}z = \int \frac{\mathrm{d}z}{1 - z} - \int \frac{\mathrm{d}z}{(1-z)^2} = \frac{1}{z-1} - \ln|z - 1| + C$$

Therefore, the implicit solution is

$$\ln|y - x| - \frac{x}{y - x} = C, \quad C \in \mathbb{R},$$

complemented by the additional solution curve $y = x$.

(iii) Substituting $z = y/x$ into the equation gives

$$\begin{aligned}(x^2 + y^2)\,\mathrm{d}x + 3xy\,\mathrm{d}y = (x^2 + (xz)^2)\,\mathrm{d}x + 3x^2z(z\,\mathrm{d}x + x\,\mathrm{d}z) = 0,\\ \frac{\mathrm{d}z}{\mathrm{d}x} = -\frac{1 + 4z^2}{3xz}.\end{aligned}$$

This is a separable ODE without constant solutions, hence equivalent to

$$\begin{aligned}\int \frac{z}{1 + 4z^2}\,\mathrm{d}z + \int \frac{1}{3x}\,\mathrm{d}x = 0,\\ \frac{1}{3}\ln|x| + \frac{1}{8}\ln\left|1 + 4z^2\right| = C,\\ 8\ln|x| + 3\ln\left|1 + 4z^2\right| = C',\\ x^8\left(1 + 4y^2/x^2\right)^3 = \mathrm{e}^{C'},\\ x^2(x^2 + 4y^2)^3 = C'', \quad C'' > 0. \qquad \text{(S)}\end{aligned}$$

This is the desired implicit solution, except for the integral curve $x = 0$, which is a solution of $(x^2 + y^2)\,\mathrm{d}x + 3xy\,\mathrm{d}y = 0$ (check it!) but missed by the substitution $z = y/x$, as mentioned earlier. However, if we allow in (S) also $C'' = 0$ then this solution is included.

(iv) This ODE is not homogeneous, but it can be transformed into a homogeneous ODE by a translation $x = u + a$, $y = v + b$. Since $x - y - 1 = x + 4y - 6 = 0$ has the solution $(x, y) = (2, 1)$, the point $(2, 1)$ is singular and the corresponding translation is $x = u + 2$, $y = v + 1$ (clear, since it removes the constants). Using $\mathrm{d}x = \mathrm{d}u$, $dy = \mathrm{d}v$, the transformed ODE is then

$$\big(u + 2 - (v + 1) - 1\big)\,\mathrm{d}u + \big(u + 2 + 4(v + 1) - 6\big) = (u - v)\,\mathrm{d}u + (u + 4v)\,\mathrm{d}v = 0.$$

Substituting $z = v/u$ into the equation gives

$$\begin{aligned}(1 - z)\,\mathrm{d}u + (1 + 4z)(z\,\mathrm{d}u + u\,\mathrm{d}z) &= 0,\\ (4z^2 + 1)\,\mathrm{d}u + u(1 + 4z)\,\mathrm{d}z &= 0,\\ \frac{\mathrm{d}z}{\mathrm{d}u} &= -\frac{1 + 4z^2}{u(1 + 4z)}.\end{aligned}$$

This is a separable ODE without constant solutions and hence equivalent to

$$\begin{aligned}\int \frac{1}{u}\,\mathrm{d}u + \int \frac{1 + 4z}{1 + 4z^2}\,\mathrm{d}z &= 0\\ ln|u| + \frac{1}{2}\ln\left(1 + 4z^2\right) + \frac{1}{2}\arctan(2z) &= C,\\ 2\ln|u| + \ln\left(1 + 4z^2\right) + \arctan(2z) &= 2C,\\ \ln(u^2 + 4v^2) + \arctan(2v/u) &= 2C,\\ \ln\big((x - 2)^2 + 4(y - 1)^2\big) + \arctan\left(\frac{2(y - 1)}{x - 2}\right) &= 2C = C', \quad C' \in \mathbb{R}.\end{aligned}$$

This is the desired implicit solution.

21 a) The given hint complicates things and shouldn't be followed. (The equation becomes more complicated, and every such circle has two points on the y-axis, so that one of them needs to fixed in advance.) I am sorry for getting this wrong.

It is better to parametrize in such a way that the midpoint has coordinates $(0, C)$, in which case the equation is $x^2 + (y - C)^2 = C^2 + 1$, or $x^2 + y^2 = 1 + 2Cy$. Solving for C gives $f(x, y) := \frac{x^2+y^2-1}{2y} = C$. It follows that an ODE for the circles is

$$f_x\,\mathrm{d}x + f_y\,\mathrm{d}y = \frac{x}{y}\,\mathrm{d}x + \frac{y^2 - x^2 + 1}{2y^2}\,\mathrm{d}y = 0.$$

The orthogonal trajectories then solve the ODE

$$-f_y\,\mathrm{d}x + f_x\,\mathrm{d}y = \frac{x^2 - y^2 - 1}{2y^2}\,\mathrm{d}x + \frac{x}{y}\,\mathrm{d}y = 0.$$

Clearing the denominator gives the equivalent form

$$(x^2 - y^2 - 1)\,\mathrm{d}x + 2xy\,\mathrm{d}y = 0.$$

(Since the parametrized x-axis ($y = 0$) is not a solution, no new solution is introduced.) This ODE is not exact but, since $\frac{M_y - N_x}{N} = \frac{-2y-2y}{2xy} = -\frac{2}{x}$ depends only on

x, has the integrating factor $\int e^{-2/x}\,dx = \frac{1}{x^2}$. The corresponding exact equation, viz. $(1 - y^2/x^2 - 1/x^2)\,dx + (2y/x)\,dy = 0$, has antiderivative $g(x, y) = x + y^2/x + 1/x$, so that the orthogonal trajectories are given by $g(x, y) = C$ or, after clearing the denominator, $x^2 + y^2 + 1 = Cx$. To this we must add the y-axis, which is a solution ($x = 0 \wedge dx = 0$), but was lost when we applied the integrating factor. The family of circles can be rewritten as $(x - C/2)^2 + y^2 = C^2/4 - 1$. Replacing $C/2$ by C the equation becomes

$$(x - C)^2 + y^2 = C^2 - 1, \quad \text{where } |C| > 1.$$

Thus the orthogonal trajectories (except for the y-axis) are itself circles, which have their midpoints on the x-axis outside the interval $[-1, 1]$ and are contained either in the left or the right half plane. One of the intersection points of the circles with the x-axis is contained in $[-1, 1]$ and approaches the origin when $C \to \pm\infty$; see Fig. **??**. (For the circles in the right half plane, which have $C > 1$, the intersection point has x-coordinate $C - \sqrt{C^2 - 1}$.)

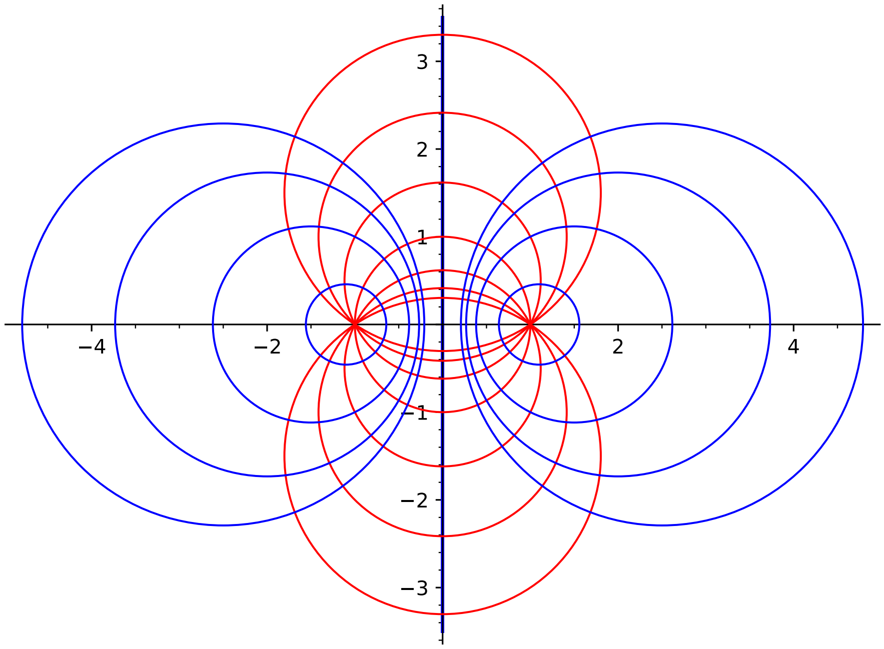


Figure 1: Orthogonal trajectories (in blue) of the family of circles through $(1, 0)$ and $(-1, 0)$

b) The focus of $y = k\,x^2 + c$ is on the y-axis and has the same y-coordinate as the points with slope ± 1 on the parabola, i.e., $y = k\left(\frac{1}{2k}\right)^2 + c = \frac{1}{4k} + c$. (At these points the parabola intersects the corresponding vertical lines in a 45° angle, so that "light beams coming from above" are reflected horizontally toward the y-axis.) Hence, if we set $c = -1/4k$ then all parabolas in the family have their focus at the origin.

$y = kx^2 - 1/4k$ is equivalent to $4k^2x^2 - 4ky - 1 = 0$, which has solutions $k = \frac{1}{8x^2}\left(4y \pm \sqrt{16y^2 + 16x^2}\right) = \frac{1}{2x^2}\left(y \pm \sqrt{x^2 + y^2}\right)$. Since $k > 0$, the parabolas are given by

$$\frac{y + \sqrt{x^2 + y^2}}{2x^2} = k, \quad k > 0.$$

Working with this function, however, leads to an elaborate computation. (I have given up it at some point and tried to find an alternative way.) The following observation helps to simplify things:

$$\frac{y+\sqrt{x^2+y^2}}{2x^2} = \frac{(y+\sqrt{x^2+y^2})(-y+\sqrt{x^2+y^2})}{2x^2(-y+\sqrt{x^2+y^2})} = \frac{1}{2}\,\frac{1}{\sqrt{x^2+y^2}-y}.$$

Hence, replacing k by $C = 1/2k$, the parabolas are also given by

$$f(x,y) := \sqrt{x^2+y^2} - y = C, \quad C > 0.$$

The corresponding differential equation is

$$\frac{x}{\sqrt{x^2+y^2}}\,\mathrm{d}x + \left(\frac{y}{\sqrt{x^2+y^2}} - 1\right)\mathrm{d}y = 0,$$

so that (after clearing denominators) the orthogonal trajectories solve

$$\left(\sqrt{x^2+y^2} - y\right)\mathrm{d}x + x\,\mathrm{d}y = 0.$$

This ODE is homogeneous with corresponding explicit form

$$y' = \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y-\sqrt{x^2+y^2}}{x} = \frac{y}{x} \pm \sqrt{1+\left(\frac{y}{x}\right)^2},$$

with the plus sign holding in the half plane $x < 0$. The usual substitution $z = y/x$ turns it into the separable ODE

$$z' = \pm\frac{\sqrt{1+z^2}}{x}.$$

Using $\int \frac{1}{\sqrt{1+z^2}}\,\mathrm{d}z = \operatorname{arsinh} z = \ln\bigl(z+\sqrt{1+z^2}\bigr)$, we obtain

$$\begin{aligned} \ln\bigl(z+\sqrt{1+z^2}\bigr) &= \pm\ln|x| + C \\ z+\sqrt{1+z^2} &= \mathrm{e}^C\,|x|^{\pm 1}, \\ \frac{y}{x} + \sqrt{1+\left(\frac{y}{x}\right)^2} = \mathrm{e}^C\,|x|^{\pm 1} &= \begin{cases} \mathrm{e}^C/x & \text{for } x > 0, \\ -\mathrm{e}^C x & \text{for } x < 0. \end{cases} \end{aligned}$$

For $x > 0$ this turns into $y + \sqrt{x^2+y^2} = \mathrm{e}^C$, and for $x < 0$ into

$$\frac{y}{x^2} + \frac{1}{x}\sqrt{\frac{x^2+y^2}{x^2}} = \frac{y-\sqrt{x^2+y^2}}{x^2} = -\mathrm{e}^C.$$

Multiplying numerator and denominator with $y + \sqrt{x^2+y^2}$ (same trick as above) turns the latter into $\frac{1}{y+\sqrt{x^2+y^2}} = \mathrm{e}^C$ or, equivalently, $y + \sqrt{x^2+y^2} = \mathrm{e}^{-C}$. Since $\mathrm{e}^{\pm C}$ can be any positive real number, the distinction of the two cases is artificial and the orthogonal trajectories of the original family of parabolas is simply

$$\sqrt{x^2+y^2} + y = C, \quad C > 0.$$

Inspecting the two representations $\sqrt{x^2+y^2} \pm y = C$, we see that the 2nd family is simply the mirror image of the 1st family with respect to the x-axis. It can also be specified as $y = k\,x^2 - \frac{1}{4k}$, $k < 0$.

In the earlier Calculus III exercise these families, with the roles of x and y interchanged, arose as images of the coordinate lines $x =$ const. and $y =$ const. under the complex squaring map $z \mapsto z^2$. Their orthogonality can also be explained by the fact that the squaring map is conformal.

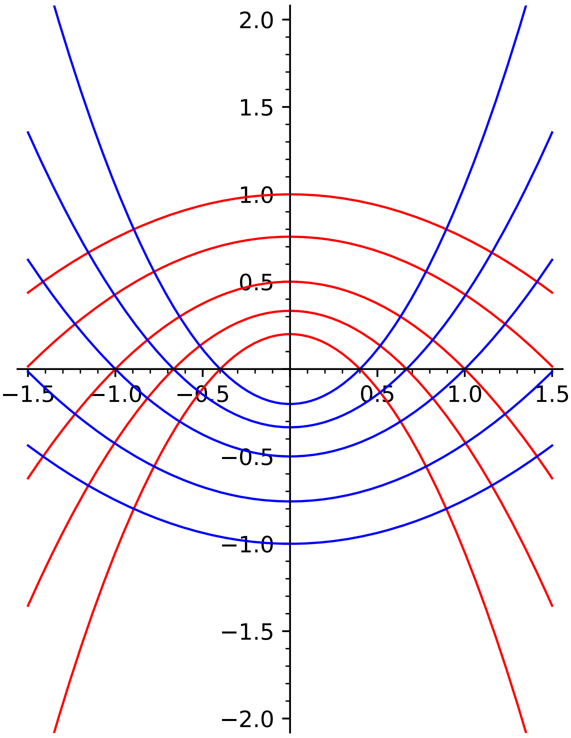


Figure 2: Orthogonal trajectories (in blue) of the family of parabolas $y = k\,x^2 - \frac{1}{4k}$, $k > 0$ (which turn out to be their mirror images with respect to the x-axis)

22 This problem is also discussed in [BDM17], Ch. 2.5, Exercise 19.
According to the model $dy/dt = ay - by^2 - Ey = (a-E)y - by^2$,

$$\Delta = (a-E)^2 \geqslant 0$$

1) When $E > a$, we have $y_1 = (a-E)/b < 0$, $y_2 = 0$.
   If the initial population $y_0 = y(t_0)$ is a positive number, then $\lim_{t\to\infty} y(t) = 0$; cf. the discussion of the harvesting equation in the lecture. In this case, the harvesting is not sustainable.

2) When $E = a$, we have $y_1 = y_2 = 0$.
   Again this implies $\lim_{t\to\infty} y(t) = 0$ if $y_0 > 0$, so the harvesting is not sustainable either.

3) When $E < a$, we have $y_1 = 0$, $y_2 = (a-E)/b > 0$. If $y_0 > 0$, then in the long run

$$\lim_{t\to\infty} y(t) = y_2 = \frac{a-E}{b}.$$

Hence,

$$\lim_{t\to\infty}\big(E\,y(t)\big) = \frac{E(a-E)}{b}$$

The eventual yield in this case, $Y = E\,y_2 = E(a-E)/b$, defines a parabola. Therefore, Y is maximized when $E = a/2$, with maximum value $a^2/4b$.

Therefore, E should equal $a/2$ in order to maximize the yield Ey in the long run.

23 First a remark on the cases $L = \pm\infty$. If (x_n) is unbounded then (and only then) for every $R \in \mathbb{R}$ there exist infinitely many indexes n such that $x_n > R$, and hence it is natural to call $+\infty$ an accumulation point of (x_n) and set $L = +\infty$ in this case. On the other hand, if (x_n) diverges to $-\infty$ then (and only then) for every $R \in \mathbb{R}$ there exist only finitely many indexes n such that $x_n > R$, but of course infinitely many indexes n such that $x_n < R$, and hence it is natural to call $-\infty$ an accumulation point of (x_n) and set $L = -\infty$ in this case, since there is no other accumulation point. The case $L = -\infty$ doesn't occur for nonnegative sequences like $x_n = \sqrt[n]{|a_n|}$.

a) Suppose the power series converges for some $z_1 \neq a$ and set $r = |z_1 - a|$, which is then > 0. Since $\sum a_n(z_1-a)^n$ converges, there exists a constant $M > 1$ such that $|a_n(z_1-a)^n| = |a_n|\,r^n \leq M$ for all n. Hence

$$\sqrt[n]{|a_n|} \leq \frac{\sqrt[n]{M}}{r} \leq \frac{M}{r} \quad \text{for all } n,$$

contradicting the unboundedness of $\sqrt[n]{|a_n|}$.

b) Assume $\lim_{n\to\infty}\sqrt[n]{|a_n|} = 0$. Then for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $|a_n| < \epsilon^n$ for $n > N$. Now let $z \in \mathbb{C}\setminus\{a\}$ be arbitary and $r = |z-a|$, i.e., $|a_n(z-a)^n| = |a_n|\,r^n$. Setting $\epsilon = 1/(2r)$ and denoting by N the corresponding response, we get

$$|a_n|\,r^n \leq \left(\frac{1}{2r}\right)^n r^n = \frac{1}{2^n} \quad \text{for } n > N.$$

Since $\sum 2^{-n}$ converges, the series $\sum a_n(z-a)^n$ converges absolutely by the comparison test. In particular $\sum a_n(z-a)^n$ converges for all $z \in \mathbb{C}$ (including $z = a$, of course).

c) Suppose first that $z \neq a$ satisfies $r = |z-a| < 1/L$. Then $L < 1/r$, and hence there exist $\theta \in (0,1)$ and $N \in \mathbb{N}$ such that $\sqrt[n]{|a_n|} \leq \theta/r$ for all $n > N$. (The number θ need only satisfy $L < \theta/r < 1/r$, i.e., $\theta \in (rL, 1)$. Then there can be only finitely many n such that $\sqrt[n]{|a_n|} > \theta/r$.) From this we obtain $|a_n|\,r^n \leq \theta^n$ for $n > N$ and can use the comparison test with the convergent series $\sum \theta^n$ to conclude that $\sum_{n=0}^{\infty} a_n(z-a)^n$ converges.

Next suppose $r = |z-a| > 1/L$. Then $1/r < L$, and hence $\sqrt[n]{|a_n|} > 1/r$ for infinitely many n. Thus $|a_n|\,r^n > 1$ for infinitely many n, implying the divergence of $\sum_{n=0}^{\infty} a_n(z-a)^n$. (Since convergence requires $|a_n|\,r^n \to 0$.)

24 a) For $s \notin \{0,1,2,\dots\}$ we have $\binom{s}{n} \neq 0$ for all n and

$$\frac{\binom{s}{n}}{\binom{s}{n+1}} = \frac{n+1}{s-n} = \frac{1+1/n}{s/n-1} \to -1 \quad \text{for } n \to \infty.$$

$\Longrightarrow R = \lim_{n\to\infty}\left|\binom{s}{n}\big/\binom{s}{n+1}\right| = 1$ (ratio test).

b) The ODE $y' = \frac{s}{1+x}\,y$ is 1st-order linear, and hence all associated IVP's have a unique solution. Since $\mathrm{B}_s(0) = \binom{s}{0} = 1$, we must have $\mathrm{B}_s(x) = (1+x)^s$ for $-1 < x < 1$, provided we can show that $x \mapsto \mathrm{B}_s(x)$ solves $y' = \frac{s}{1+x}\,y$ as well.

For $|z| < 1$ and hence in particular for $-1 < x < 1$ we can differentiate $\mathrm{B}_s(x)$ termwise:

$$\mathrm{B}_s'(x) = \sum_{n=1}^{\infty} n\binom{s}{n}x^{n-1} = \sum_{n=1}^{\infty} s\binom{s-1}{n-1}x^{n-1}.$$

$$\begin{aligned}
\implies (1+x)\mathrm{B}_s'(x) &= \sum_{n=1}^{\infty} s\binom{s-1}{n-1}x^{n-1}(1+x) = s\left(\sum_{n=1}^{\infty}\binom{s-1}{n-1}x^{n-1} + \sum_{n=1}^{\infty}\binom{s-1}{n-1}x^{n}\right)\\
&= s\left(\sum_{n=0}^{\infty}\binom{s-1}{n}x^{n} + \sum_{n=0}^{\infty}\binom{s-1}{n-1}x^{n}\right)\\
&= s\sum_{n=0}^{\infty}\left[\binom{s-1}{n-1} + \binom{s-1}{n}\right]x^{n}\\
&= s\sum_{n=0}^{\infty}\binom{s}{n}x^{n} = s\,\mathrm{B}_s(x),
\end{aligned}$$

as claimed.

c) The computation in b) is valid for all $z \in \mathbb{C}$ with $|z| < 1$, showing that $\mathrm{B}_s'(z) = \frac{s}{1+z}\,\mathrm{B}_s(z)$ for such z. The extension of the chain rule to complex differentiation (proved as in the real case) gives

$$\frac{\mathrm{d}}{\mathrm{d}z}\,\mathrm{e}^{s\log(1+z)} = \mathrm{e}^{s\log(1+z)}\,\frac{\mathrm{d}}{\mathrm{d}z}\,[s\log(1+z)] = \mathrm{e}^{s\log(1+z)}\,\frac{s}{1+z}.$$

Thus both functions are solutions of $y' = \frac{s}{1+z}\,y$, $y(0) = 1$. For the uniqueness proof we consider the function $f(z) = \mathrm{B}_s(z)\,\mathrm{e}^{-s\log(1+z)}$, defined for $|z| < 1$. Using the product rule for differentiation of complex functions, we have

$$\begin{aligned}
f'(z) &= \mathrm{B}_s'(z)\,\mathrm{e}^{-s\log(1+z)} + \mathrm{B}_s(z)\,\frac{\mathrm{d}}{\mathrm{d}z}\,\mathrm{e}^{-s\log(1+z)}\\
&= \frac{s}{1+z}\mathrm{B}_s(z)\,\mathrm{e}^{-s\log(1+z)} + \mathrm{B}_s(z)\,\mathrm{e}^{-s\log(1+z)}\,\frac{-s}{1+z} = 0,
\end{aligned}$$

and of course $f(0) = 1$. This implies $f(z) \equiv 1$ (since, e.g., $\mathrm{Re}f$ and $\mathrm{Im}f$ are real 2-variable functions with vanishing differential and hence must be constant), and hence $\mathrm{B}_s(z) = \mathrm{e}^{s\log(1+z)}$ for $|z| < 1$.

Differential Equations (Math 285)

H25 Evaluate the two series

$$\sum_{n=1}^{\infty}\frac{\sin(nx)}{n^3} \quad \text{and} \quad \sum_{n=1}^{\infty}\frac{\cos(nx)}{n^4} \quad \text{for } x\in\mathbb{R},$$

in a way similar to the evaluation of $\sum_{n=1}^{\infty}\frac{\cos(nx)}{n^2}$ in the lecture, and use this in turn to evaluate the series

$$1-\frac{1}{3^3}+\frac{1}{5^3}-\frac{1}{7^3}+\frac{1}{9^3}-\frac{1}{11^3}\pm\cdots,$$
$$1+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\frac{1}{5^4}+\frac{1}{6^4}+\cdots$$

H26

a) Assuming that $\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$, show that $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2}=\frac{\pi^2}{12}$ without resorting to the evaluation of $\sum_{n=1}^{\infty}\frac{\cos(nx)}{n^2}$.
Hint: Add the two series.

b) Show that $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\cdots=\frac{\pi^2}{8}$.

H27 Solve the initial value problem

$$y''+|y|=0, \quad y(0)=0,\ y'(0)=1.$$

Your solution should have the (maximal) domain $\mathbb{R}$. Is the solution unique?
Hint: The solution of Example 10 in `lecture1-3_handout.pdf` and Exercise H29 a) may help.

H28 Let (M,d) be a metric space and $(a,b)\in M\times M$.

a) Show that the metric d is *continuous* in the following sense:
For every $\epsilon>0$ there exists $\delta>0$ such that $d(x,a)<\delta\wedge d(y,b)<\delta$ implies $|d(x,y)-d(a,b)|<\epsilon$.
Hint: First derive the so-called *quadrangle inequality* $|d(x,y)-d(a,b)|\leq d(x,a)+d(y,b)$.

b) Using a), show in detail that $x_n\to a$ and $y_n\to b$ implies $d(x_n,y_n)\to d(a,b)$. (A special case of this, viz. $d(x_n,b)\to d(a,b)$, is used in the proof of Part (2) of Banach's Fixed-Point Theorem.)

H29 a) Show that the general (real) solution of $y'' = y$ is $y(x) = c_1 e^x + c_2 e^{-x}$, $c_1, c_2 \in \mathbb{R}$.

Hint: For a solution y the functions $y + y'$ and $y - y'$ satisfy linear 1st-order ODE's.

b) For $x \in \mathbb{R}$ let

$$F(x) = \int_0^\infty \frac{\cos(xt)}{t^2+1}\,\mathrm{d}t\,.$$

Show that

$$F'(x) = -\frac{\pi}{2} + \int_0^\infty \frac{\sin(xt)}{t(t^2+1)}\,\mathrm{d}t \quad \text{for } x > 0.$$

Hint: Differentiate F under the integral sign and use $\int_0^\infty \sin(xt)/t\,\mathrm{d}t = \int_0^\infty \sin(t)/t\,\mathrm{d}t = \pi/2$ for $x > 0$.

c) Show that F solves $y'' = y$ on $(0, \infty)$.

d) Determine F from a), c) and $F(0)$, $F'(0+)$, and use the result to evaluate the integral

$$\int_0^\infty \frac{\cos t}{t^2+1}\,\mathrm{d}t\,.$$

Due on Thu Mar 21, 10 am

Metric spaces (required for Exercise H28) will be discussed in more detail in the lecture on Wed Mar 20, but H28 can be solved by only using the axioms for a metric space listed on Slide 23 of `lecture15-17_handout.pdf`.

Exercises H29 b)–d) are optional, but should be handed in together with H29 a) on Mar 21. Rigorous justifications likely require that you study the material on uniform convergence of improper parameter integrals in `lecture11-13_handout.pdf`, which was skipped in the Math 285 lecture.

Solutions

25 The two series converge uniformly on $\mathbb{R}$ due to the inequalities

$$\left|\frac{\sin(nx)}{n^3}\right| \leqslant \frac{1}{n^3},$$
$$\left|\frac{\cos(nx)}{n^4}\right| \leqslant \frac{1}{n^4},$$

and the known fact that $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent when $p > 1$. But for termwise differentiability, which we want to employ, we need only the pointwise convergence and the uniform convergence of the series of derivatives.
Now let us evaluate the first series. Its series of derivatives is

$$\sum_{n=1}^{\infty} \frac{\mathrm{d}}{\mathrm{d}x} \frac{\sin(nx)}{n^3} = \sum_{n=1}^{\infty} \frac{\cos(nx)}{n^2}$$

From the lecture slides, this series converges uniformly and evaluates to

$$\sum_{n=1}^{\infty} \frac{\cos(nx)}{n^2} = \frac{(x-\pi)^2}{4} - \frac{\pi^2}{12}$$

Hence we can apply the Differentiation Theorem to conlcude that

$$\frac{\mathrm{d}}{\mathrm{d}x} \sum_{n=1}^{\infty} \frac{\sin(nx)}{n^3} = \sum_{n=1}^{\infty} \frac{\mathrm{d}}{\mathrm{d}x} \frac{\sin(nx)}{n^3} = \sum_{n=1}^{\infty} \frac{\cos(nx)}{n^2} = \frac{(x-\pi)^2}{4} - \frac{\pi^2}{12}.$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{\sin(nx)}{n^3} = \int \left(\frac{(x-\pi)^2}{4} - \frac{\pi^2}{12} \right) \mathrm{d}x = \frac{(x-\pi)^3}{12} - \frac{\pi^2 x}{12} + C$$

The constant C can be determined by setting $x = 0$:

$$0 = \sum_{n=1}^{\infty} \frac{\sin(n\,0)}{n^3} = \frac{(-\pi)^3}{12} + C \implies C = \frac{\pi^3}{12}.$$

Hence, the first series can be expressed as

$$\sum_{n=1}^{\infty} \frac{\sin(nx)}{n^3} = \frac{(x-\pi)^3}{12} - \frac{\pi^2 x}{12} + \frac{\pi^3}{12}.$$

Plugging in $x = \pi/2$ yields, on account of $\sin\big(2k(\pi/2)\big) = \sin(k\pi) = 0$, $\sin\big((2k+1)\pi/2\big) = (-1)^k$,

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^3} = \sum_{n=1}^{\infty} \frac{\sin(n\pi/2)}{n^3} = \frac{(\pi/2-\pi)^3}{12} - \frac{\pi^2(\pi/2)}{12} + \frac{\pi^3}{12} = \pi^3 \left(-\frac{1}{96} - \frac{1}{24} + \frac{1}{12} \right)$$
$$= \frac{\pi^3}{32}.$$

The series of derivatives of the second series, up to sign, is exactly this series. As shown at the beginning, $\sum_{n=1}^{\infty} \sin(nx)/n^3$ converges uniformly on $\mathbb{R}$. Hence we can apply the Differentiation Theorem again and conclude that

$$\frac{\mathrm{d}}{\mathrm{d}x} \sum_{n=1}^{\infty} \frac{\cos(nx)}{n^4} = \sum_{n=1}^{\infty} \frac{\mathrm{d}}{\mathrm{d}x} \left[\frac{\cos(nx)}{n^4} \right] = \sum_{n=1}^{\infty} \left[-\frac{\sin(nx)}{n^3} \right] = -\frac{(x-\pi)^3}{12} + \frac{\pi^2 x}{12} - \frac{\pi^3}{12}.$$

Hence,

$$\sum_{n=1}^{\infty} \frac{\cos(nx)}{n^4} = -\frac{(x-\pi)^4}{48} + \frac{\pi^2 x^2}{24} - \frac{\pi^3 x}{12} + B, \quad B \in \mathbb{R}.$$

In order to determine the constant B, we evaluate the integral of this function over a full period in two ways (as in the lecture). Using the Integration Theorem, we have

$$\int_0^{2\pi} \sum_{n=1}^{\infty} \frac{\cos(nx)}{n^4} \,\mathrm{d}x = \sum_{n=1}^{\infty} \int_0^{2\pi} \frac{\cos(nx)}{n^4} \,\mathrm{d}x = 0$$

. On the other hand,

$$\begin{aligned}
\int_0^{2\pi} \sum_{n=1}^{\infty} \frac{\cos(nx)}{n^4} \,\mathrm{d}x &= \int_0^{2\pi} \left(-\frac{(x-\pi)^4}{48} + \frac{\pi^2 x^2}{24} - \frac{\pi^3 x}{12} + B \right) \mathrm{d}x \\
&= \left[-\frac{(x-\pi)^5}{240} + \frac{\pi^2 x^3}{72} - \frac{\pi^3 x^2}{24} + Bx \right]_{x=0}^{2\pi} \\
&= \left(-\frac{\pi^5}{240} + \frac{\pi^2 (2\pi)^3}{72} - \frac{\pi^3 (2\pi)^2}{24} + B(2\pi) - \frac{\pi^5}{240} \right) = -\frac{46\,\pi^5}{720} + B(2\pi) = 0.
\end{aligned}$$

$$\Longrightarrow B = \frac{23\pi^4}{720}$$

Hence, the first series can be expressed as

$$\sum_{n=1}^{\infty} \frac{\cos(nx)}{n^4} = -\frac{(x-\pi)^4}{48} + \frac{\pi^2 x^2}{24} - \frac{\pi^3 x}{12} + \frac{23\pi^4}{720}.$$

Finally, we evaluate the series $\sum_{n=1}^{\infty} 1/n^4$.
Substituting $x = 0$ into the second series, we obtain

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \sum_{n=1}^{\infty} \frac{\cos(n\,0)}{n^4} = -\frac{\pi^4}{48} + \frac{23\pi^4}{720} = \frac{\pi^4}{90}.$$

Remark: Continuing in this way, one can obtain closed-form expressions for $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(nx)}{n^{2p-1}}$ and $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(nx)}{n^{2p}}$ for all positive integers p, and use this to evaluate the series $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{2p-1}}$ $(p = 2, 3, 4, \dots)$ and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{2p}}$ $(p = 1, 2, 3, \dots)$. The answers are of the form $a_p\,\pi^{2p-1}$, respectively, $b_p\,\pi^{2p}$ with certain numbers $a_p, b_p \in \mathbb{Q}$ ($\to$ Euler and Bernoulli numbers). About the values of the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{2p-1}}$ $(p = 2, 3, 4, \dots)$ much less is known. (Essentially the only thing known is that for $p = 2$ the value is an irrational number.)

26 a) Using the hint and subtracting the two series, we obtain

$$\sum_{n=1}^{\infty}\frac{1}{n^2}-\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2}=2\left(\frac{1}{2^2}+\frac{1}{4^2}+\frac{1}{6^2}+\frac{1}{8^2}+\cdots\right)$$
$$=\frac{1}{2}\left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots\right)=\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n^2}.$$
$$\Longrightarrow\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2}=\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{12}.$$

b) Adding the two series, we obtain

$$\sum_{n=1}^{\infty}\frac{1}{n^2}+\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2}=2\left(\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\cdots\right)=\frac{\pi^2}{6}+\frac{\pi^2}{12}=\frac{\pi^2}{4}.$$
$$\Longrightarrow\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\cdots=\frac{\pi^2}{8}$$

27 [Note: In the following, the general solutions of the ODE's $y''\pm y=0$ are determined using the machinery for higher-order linear ODE's with constant coefficients. You will learn this stuff during the next few weeks. Adhoc derivations of these solutions were given in `lecture1-3_handout.pdf`, Slides 35 f and Exercise H29 a).]
When $y\geq 0$, $y''+y=0$. The characteristic polynomial is $X^2+1=0$ with roots $\lambda_1=i,\lambda_2=-i$.
The general real solution is $y(t)=c_1\cos t+c_2\sin t$, $c_1,c_2\in\mathbb{R}$.

$$\because\begin{cases}y(0)=c_1=0\\ y'(0)=c_2=1\end{cases}\qquad\therefore y=\sin t\quad(t\in[0,\pi])$$

When $y\leq 0$, $y''-y=0$. The characteristic polynomial is $X^2-1=0$ with roots $\lambda_1=1,\lambda_2=-1$.
$\Longrightarrow$ The general real solution is $y(t)=c_1e^t+c_2e^{-t}$, $c_1,c_2\in\mathbb{R}$.

$$\because\begin{cases}y(0)=c_1+c_2=0\\ y'(0)=c_1-c_2=1\end{cases}\Longrightarrow\begin{cases}c_1=\frac{1}{2}\\ c_2=-\frac{1}{2}\end{cases}\quad\therefore y(t)=\frac{1}{2}e^t-\frac{1}{2}e^{-t}\quad(t\leq 0)$$

In order to get maximal domain $\mathbb{R}$, we impose for $t\geq\pi$ the new initial conditions $y(\pi)=\sin(\pi)=0$, $y'(\pi)=\cos(\pi)=-1$, which are satisfied by the already defined solution on $(-\infty,\pi]$. Since $y(t)<0$ for $t\downarrow\pi$, we must fit the general solution for $y\leq 0$, viz. $y(t)=c_1e^t+c_2e^{-t}$, to the new initial conditions.

$$\because\begin{cases}y(\pi)=c_1e^{\pi}+c_2e^{-\pi}=0\\ y'(\pi)=c_1e^{\pi}-c_2e^{-\pi}=-1\end{cases}\Rightarrow\begin{cases}c_1=-\frac{1}{2e^{\pi}}\\ c_2=\frac{1}{2e^{-\pi}}\end{cases}\quad\therefore y(t)=-\frac{1}{2e^{\pi}}\,\mathrm{e}^t+\frac{1}{2e^{-\pi}}\,e^{-t}\quad(t\geq\pi)$$

Since this function is negative for all $t>\pi$, it also provides a solution of $y''+|y|=0$ on $[\pi,+\infty)$.

The final solution is

$$y(t) = \begin{cases} \frac{1}{2}e^t - \frac{1}{2}e^{-t} = \sinh t & \text{for } t \le 0, \\ \sin t & \text{for } 0 \le t \le \pi, \\ -\frac{1}{2}\,\mathrm{e}^{t-\pi} + \frac{1}{2}\,e^{-(t-\pi)} = -\sinh(t-\pi) & \text{for } t \ge \pi. \end{cases}$$

The function $y(t)$ is differentiable also at $t = 0, \pi$, because the one-sided derivatives exist there and coincide. (In fact $y(t)$ is even C^2, but not C^3.)

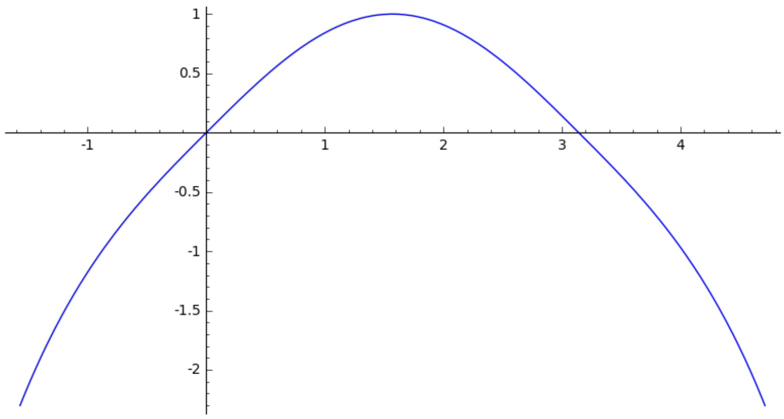


Figure 1: The solution $y(t)$ to H29

The Existence and Uniqueness Theorem applies to $y'' + |y| = 0$, because it is equivalent to the explicit ODE $y'' = f(t, y, y')$ with $f(t, y_0, y_1) = -|y_0|$. The function $f(t, y_0, y_1)$is continuous and satisfies

$$|f(t, y_0, y_1) - f(t, z_0, z_1)| = |-|y_0| + |z_0|| = \pm\big(|y_0| - |z_0|\big) \le |y_0 - z_0| \le \sqrt{(y_0 - z_0)^2 + (y_1 - z_1)^2}$$

for all $\mathbf{y} = (y_0, y_1)$, $\mathbf{z} = (z_0, z_1) \in \mathbb{R}^2$, i.e., a global Lipschitz condition with $L = 1$. As shown in the lecture, the (trivially continuous) 1st-order system obtained from $y'' = f(t, y, y')$ by order-reduction then satisfies such a Lipschitz condition as well (perhaps with slightly larger Lipschitz constant), so that the Existence and Uniqueness Theorem can be applied.

28 a) Applying the triangle inequality twice, we have

$$\begin{aligned} d(x, y) &\le d(x, a) + d(a, y) \\ &\le d(x, a) + d(a, b) + d(b, y). \\ \implies d(x, y) - d(a, b) &\le d(x, a) + d(y, b) \end{aligned}$$

Interchanging x, a as well as y, b in this inequality turns the left-hand side into $d(a, b) - d(x, y)$ and preserves the right-hand side, so that we also have $d(a, b) - d(x, y) \le d(x, a) + d(y, b)$. Thus $\pm\big(d(x, y) - d(a, b)\big) \le d(x, a) + d(y, b)$, which is equivalent to the quadrangle inequality.

With the quadrangle inequality at hand the continuity of d is easy to prove: Just choose $\delta = \epsilon/2$ as response to ϵ.

b) Let $\epsilon > 0$ be given. There exists $N_1 \in \mathbb{N}$ such that $d(x_n, a) < \epsilon/2$ for all $n > N_1$, and $N_2 \in \mathbb{N}$ such that $d(y_n, b) < \epsilon/2$ for all $n > N_2$. Using the qudrangle inequality, we then have

$$|d(x_n, y_n) - d(a, b)| \leq d(x_n, a) + d(y_n, b) < \epsilon/2 + \epsilon/2 = \epsilon$$

for all $n \geq \max\{N_1, N_2\}$. Thus $N = \max\{N_1, N_2\}$ can serve as response to ϵ in a proof of $d(x_n, y_n) \to d(a, b)$.

29 a) We have

$$(y + y')' = y' + y'' = y' + y = y + y',$$
$$(y - y')' = y' - y'' = y' - y = -(y - y'),$$

i.e., $z = y + y'$ satisfies $z' = z$ and $w = y - y'$ satisfies $w' = -w$. From the theory of 1st-order linear ODE's it follows that $z(x) = y(x) + y'(x) = c_1 \mathrm{e}^x$, $w(x) = y(x) - y'(x) = c_2 \mathrm{e}^{-x}$ for some $c_1, c_2 \in \mathbb{R}$. $\implies y(x) = \frac{1}{2}(c_1 \mathrm{e}^x + c_2 \mathrm{e}^{-x}) = (c_1/2)\mathrm{e}^x + (c_2/2)\mathrm{e}^{-x}$, which is of the required form.

b) From the lecture recall that F is continuous on $\mathbb{R}$ and can be differentiated under the integral sign for $x > 0$. Thus for $x > 0$ we have

$$F'(x) = -\int_0^\infty \frac{t \sin(xt)}{t^2 + 1}\,\mathrm{d}t = -\int_0^\infty \frac{t^2 \sin(xt)}{t(t^2 + 1)}\,\mathrm{d}t = -\int_0^\infty \frac{(t^2 + 1 - 1)\sin(xt)}{t(t^2 + 1)}\,\mathrm{d}t$$
$$= -\int_0^\infty \frac{\sin(xt)}{t}\,\mathrm{d}t + \int_0^\infty \frac{\sin(xt)}{t(t^2 + 1)}\,\mathrm{d}t\,.$$

The first integral is actually independent of x, since

$$\int_0^\infty \frac{\sin(xt)}{t} = \int_0^\infty \frac{\sin s}{(s/x)x}\,\mathrm{d}s = \int_0^\infty \frac{\sin s}{s}\,\mathrm{d}s, \qquad \text{(Subst. } s = xt,\ \mathrm{d}s = x\,\mathrm{d}t)$$

and has the value $\pi/2$, as we know from the Calculus III fnal exam.

c) Differentiating the expression in b) again under the integral sign, we obtain

$$F''(x) = \int_0^\infty \frac{\mathrm{d}}{\mathrm{d}x} \frac{\sin(xt)}{t(t^2 + 1)}\,\mathrm{d}t = \int_0^\infty \frac{t \cos(xt)}{t(t^2 + 1)}\,\mathrm{d}t = \int_0^\infty \frac{\cos(xt)}{t^2 + 1}\,\mathrm{d}t = F(x).$$

This is justified, since

$$\left|\frac{\mathrm{d}}{\mathrm{d}x} \frac{\sin(xt)}{t(t^2 + 1)}\right| = \frac{|\cos(xt)|}{t^2 + 1} \leq \frac{1}{t^2 + 1} = \Phi(t),$$

which is independent of x and integrable over $(0, \infty)$.

d) According to a) and c) we have

$$F(x) = c_1 \mathrm{e}^x + c_2 \mathrm{e}^{-x},$$
$$F'(x) = c_1 \mathrm{e}^x - c_2 \mathrm{e}^{-x}$$

for some $c_1, c_2 \in \mathbb{R}$ and $x > 0$. Since F is continuous in 0, the first identity holds also for $x = 0$ and gives $c_1 + c_2 = F(0) = \int_0^\infty \frac{\mathrm{d}t}{t^2+1} = \pi/2$.

Since

$$\left|\frac{\sin(xt)}{t(t^2+1)}\right| \leq \frac{1}{t(t^2+1)} = \Phi(t),$$

which is independent of x and integrable over $(0, \infty)$, we get

$$F'(0+) = -\frac{\pi}{2} + \int_0^\infty \lim_{x\downarrow 0} \frac{\sin(xt)}{t(t^2+1)}\,\mathrm{d}t = -\frac{\pi}{2} + \int_0^\infty 0\,\mathrm{d}t = -\frac{\pi}{2}$$

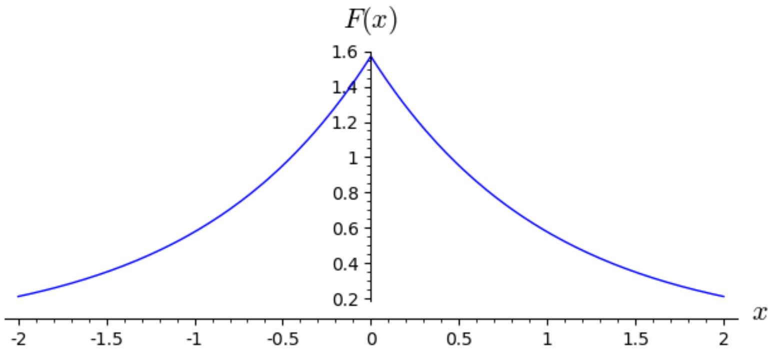
On the other hand, $F'(0+) = \lim_{x\downarrow 0}(c_1\mathrm{e}^x - c_2\mathrm{e}^{-x}) = c_1 - c_2$, so that $c_1 - c_2 = -\pi/2$. It follows that $c_1 = 0$, $c_2 = \pi/2$. Hence $F(x) = (\pi/2)\mathrm{e}^{-x}$ for $x \geq 0$ and

$$\int_0^\infty \frac{\cos t}{t^2+1}\,\mathrm{d}t = F(1) = \frac{\pi}{2\mathrm{e}}.$$

Remarks: This exercise is based on a video from the Youtube channel "Flammable Maths", who's author Jens Fehlau has shot several nice videos with quite nontrivial evaluations of interesting integrals.
Since F is even, we have $F(x) = (\pi/2)\mathrm{e}^{-|x|}$ for $x \in \mathbb{R}$. At $x = 0$ the function F is not differentiable, although the right-hand side of the integral representation

$$F'(x) = -\int_0^\infty \frac{t\sin(xt)}{t^2+1}\,\mathrm{d}t, \qquad \text{valid for } x \neq 0,$$

evaluates to zero at $x = 0$.



Numerically, $\pi/(2\mathrm{e}) \approx 0.5778636748954609$. This differs only slightly from the Euler-Mascheroni constant $\gamma = \lim_{n\to\infty}(1 + 1/2 + 1/3 + \cdots + 1/n - \ln(n)) \approx 0.5772156649015329$, so that perhaps someone who computes the integral $\int_0^\infty \frac{\cos t}{t^2+1}\,\mathrm{d}t$ numerically but doesn't know about the exact evaluation is mislead to conjecture that it has the value γ.

Differential Equations (Math 285)

H30 a) Show that a closed subset N of a complete metric space (M,d) is complete in the induced metric $d|_N\colon N\times N\to\mathbb{R}$, $(x,y)\mapsto d(x,y)$.

b) Conversely, show that a subset of a metric space that is complete in the induced metric must be closed.

H31 a) Compute the norms $\|\mathbf{A}\|$ of the following matrices $\mathbf{A}\in\mathbb{R}^{2\times 2}$ directly from the definition and compare them with their Frobenius norms $\|\mathbf{A}\|_{\mathrm{F}}$.

$$\begin{pmatrix}2&2\\2&2\end{pmatrix},\quad\begin{pmatrix}2&0\\0&-3\end{pmatrix},\quad\begin{pmatrix}\frac{1}{2}&\pm 1\\0&\frac{1}{2}\end{pmatrix}.$$

b) Show that the norm $\|\mathbf{D}\|$ of a diagonal matrix $\mathbf{D}\in\mathbb{R}^{n\times n}$ is the largest absolute value of an entry on the diagonal.

c) Show that if $T\colon\mathbb{R}^n\to\mathbb{R}^n$ is continuously differentiable with $\mathbf{J}_T(\mathbf{x}^*)=\mathbf{0}$, there exists $r>0$ such that $T\colon\overline{\mathrm{B}_r(\mathbf{x}^*)}\to\overline{\mathrm{B}_r(\mathbf{x}^*)}$ forms a contraction with constant $C=\frac{1}{2}$.

d) Give an example of a linear map $T\colon\mathbb{R}^2\to\mathbb{R}^2$, $\mathbf{x}\mapsto\mathbf{A}\mathbf{x}$ that forms a contraction but has Frobenius norm >1.

H32 For the following ODE's $y'=f(y)$, use the Existence and Uniqueness Theorem to determine the points $(t_0,y_0)\in\mathbb{R}^2$ such that the initial value problem $y'=f(y)\wedge y(t_0)=y_0$ has a unique solution near (t_0,y_0). Then solve the ODE, sketch the integral curves, and compare with your prediction.

a) $y'=|y|$; b) $y'=\sqrt{|y-y^2|}$.

H33 Use Picard-Lindelöf iteration to compute the solution $\phi=(\phi_1,\phi_2)^{\mathsf{T}}$ of the system

$$\begin{pmatrix}y_1'\\y_2'\end{pmatrix}=\begin{pmatrix}-y_2\\y_1\end{pmatrix}$$

with initial condition $\phi(0)=(1,0)^{\mathsf{T}}$.

H34 Suppose that $f\colon\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is continuous and satisfies locally a Lipschitz condition, and that

$$f(-t,y)=-f(t,y)\quad\text{for all }(t,y)\in\mathbb{R}^2.$$

Show that any solution $\phi\colon[-r,r]\to\mathbb{R}$, $r>0$, of $y'=f(t,y)$ is its own mirror image with respect to the y-axis.

 PLEASE TURN OVER

H35 *Optional Exercise*

a) Prove that $\mathbb{R}^{n\times n} \to \mathbb{R}$, $\mathbf{A} \mapsto \|\mathbf{A}\|$ satisfies (N1)–(N4).

b) Repeat a) for the Frobenius norm $\mathbb{R}^{n\times n} \to \mathbb{R}$, $\mathbf{A} \mapsto \|\mathbf{A}\|_{\mathrm{F}}$.

c) Show that $\|\mathbf{A}\| \leq \|\mathbf{A}\|_{\mathrm{F}}$ for all matrices $\mathbf{A} \in \mathbb{R}^{n\times n}$ or, equivalently, $|\mathbf{Ax}| \leq \|\mathbf{A}\|_{\mathrm{F}}\,|\mathbf{x}|$ for all $\mathbf{A} \in \mathbb{R}^{n\times n}$ and $\mathbf{x} \in \mathbb{R}^n$.

   Hint: Use $\|\mathbf{A}\| = \max\bigl\{|\mathbf{Ax}|\,;\mathbf{x} \in \mathbb{R}^n, |\mathbf{x}| = 1\bigr\}$ and the Cauchy-Schwarz Inequality for vectors in $\mathbb{R}^n$.

d) For $\mathbf{A} \in \mathbb{R}^{n\times n}$ show $\|\mathbf{A}\| = \bigl\|\mathbf{A}^{\mathsf{T}}\bigr\|$.

e) Suppose $\mathbf{A} \in \mathbb{R}^{n\times n}$ is invertible, $\mathbf{B} = \mathbf{A}^{\mathsf{T}}\mathbf{A}$, and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ are the eigenvalues of $\mathbf{B}$. Show $\|\mathbf{A}^{-1}\| = 1/\sqrt{\lambda_n}$.

f) Using the notation in e), show $\|A\|_{\mathrm{F}} = \sqrt{\lambda_1 + \lambda_2 + \cdots + \lambda_n}$. (This yields an alternative proof of the inequality $\|\mathbf{A}\| = \sqrt{\lambda_1} \leq \|\mathbf{A}\|_{\mathrm{F}}$.)

   Hint: $\|\mathbf{A}\|_{\mathrm{F}} = \operatorname{tr}\bigl(\mathbf{A}^{\mathsf{T}}\mathbf{A}\bigr)$; cf. Math257 in Fall 2023, Exercise H56 a) of Homework 11.

Due on Thu Mar 28, 10 am

Exercise H35 can be handed in until Thu April 11, 10 am. Parts c), d) of Exercise H31 are also considered as optional but should be handed together with a), b) on Mar 28.

Solutions

30 a) If (x_n) is a Cauchy sequence in N, it is a fortiori a Cauchy sequence in M and hence converges to some $a \in M$, since (M, d) is complete. But "N closed" means that N contains all limit points of sequences in N, so $a \in N$ and (x_n) converges in $(N, d|_N)$, which is therefore complete as well.

b) Using the notation in a), let (x_n) be a sequence in N, which has a limit in M, say a. Then (x_n) must be a Cauchy sequence, and hence convergent in N, since $(N, d|_N)$ is complete. Since limits of sequences are unique (the easily proved analogue for metric spaces of Exercise W18 a) of Worksheet 6 in Calculus III, Fall 2022), this implies $a \in N$. Thus N contains all limit points of sequences in N and hence is closed.

Remarks: By the term "limit point" I mean just "limit", but some people would interpret "limit points" as "accumulation points" of not necessarily convergent sequences. In fact, since closed subsets are also characterized as subsets containing all their accumulation points, both views are admitted for this exercise.

Note that in b) the completeness of M is not required, and hence b) holds also for complete subspaces of incomplete metric spaces.

31 a) Set $\mathbf{x} = \begin{pmatrix} \sin x & \cos x \end{pmatrix}^{\mathrm{T}}$ for $x \in [0, 2\pi)$.
In what follows, all maxima are taken over $x \in [0, 2\pi)$ (or over $\mathbb{R}$, which amounts to the same).

i) The norms of $\mathbf{A} = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$ are shown below.

$$\|\mathbf{A}\| = \max\{|\mathbf{A}\mathbf{x}|\} = \max\left\{\left|\begin{pmatrix} 2\sin x + 2\cos x \\ 2\sin x + 2\cos x \end{pmatrix}\right|\right\} = \max\left\{\sqrt{2\,(2\sin x + 2\cos x)^2}\right\} = 4$$

$$\|\mathbf{A}\|_F = \sqrt{2^2 + 2^2 + 2^2 + 2^2} = 4$$

Therefore for $\mathbf{A} = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$, $\|\mathbf{A}\| = \|\mathbf{A}\|_F$.

ii) The norms of $\mathbf{A} = \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix}$ are shown below.

$$\|\mathbf{A}\| = \max\{|\mathbf{A}\mathbf{x}|\} = \max\left\{\left|\begin{pmatrix} 2\sin x \\ -3\cos x \end{pmatrix}\right|\right\} = \max\left\{\sqrt{2\,(\sin x)^2 + (-3\cos x)^2}\right\} = 3$$

$$\|\mathbf{A}\|_F = \sqrt{2^2 + 0^2 + 0^2 + 3^2} = \sqrt{13}$$

Therefore for $\mathbf{A} = \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix}$, $\|\mathbf{A}\| < \|\mathbf{A}\|_F$.

iii) The norms of $\mathbf{A} = \begin{pmatrix} \frac{1}{2} & \pm 1 \\ 0 & \frac{1}{2} \end{pmatrix}$ are shown below.

$$\|\mathbf{A}\| = \max\{|\mathbf{A}\mathbf{x}|\} = \max\left\{\left|\begin{pmatrix} \frac{1}{2}\sin x \pm \cos x \\ \frac{1}{2}\cos x \end{pmatrix}\right|\right\}$$
$$= \max\left\{\sqrt{\left(\frac{1}{2}\sin x \pm \cos x\right)^2 + \left(\frac{1}{2}\cos x\right)^2}\right\} = \sqrt{\frac{3}{4} + \frac{\sqrt{2}}{2}} = \frac{1+\sqrt{2}}{2} \approx 1.207,$$
$$\|\mathbf{A}\|_F = \sqrt{\frac{1}{2}^2 + 1^2 + 0^2 + \frac{1}{2}^2} = \frac{\sqrt{6}}{2} \approx 1.225$$

(For the former, using the Calculus I machinery one finds that $x \mapsto \left(\frac{1}{2}\sin x \pm \cos x\right)^2 + \left(\frac{1}{2}\cos x\right)^2 = \frac{1}{4} + \cos^2 x \pm \sin x \cos x$ is maximized at $x_1 = \pm\pi/8$ and $x_2 = \pm 5\pi/8$ with value $\frac{3}{4} + \frac{1}{2}\sqrt{2} = \frac{3+2\sqrt{2}}{4}$.)

Therefore for $\mathbf{A} = \begin{pmatrix} \frac{1}{2} & \pm 1 \\ 0 & \frac{1}{2} \end{pmatrix}$, $\|\mathbf{A}\| < \|\mathbf{A}\|_F$.

Remark: Here is the alternative computation using the formula $\|\mathbf{A}\| = \sqrt{\rho\left(\mathbf{A}^\mathsf{T}\mathbf{A}\right)}$:

$$\begin{aligned}
\mathbf{B} = \mathbf{A}^\mathsf{T}\mathbf{A} &= \begin{pmatrix} \frac{1}{4} & \pm\frac{1}{2} \\ \pm\frac{1}{2} & \frac{5}{4} \end{pmatrix}, \\
\chi_\mathbf{B}(X) &= X^2 - \frac{3}{2}X + \frac{5}{16} - \frac{1}{4} = X^2 - \frac{3}{2}X + \frac{1}{16}, \\
\lambda_{1/2} &= \frac{1}{2}\left(\frac{3}{2} \pm \sqrt{\frac{9}{4} - \frac{4}{16}}\right) = \frac{1}{4}\left(3 \pm 2\sqrt{2}\right) = \frac{1}{4}\left(1 \pm \sqrt{2}\right)^2, \\
\rho(\mathbf{B}) &= \lambda_1 = \frac{1}{4}\left(1+\sqrt{2}\right)^2, \\
\|\mathbf{A}\| &= \sqrt{\lambda_1} = \frac{1}{2}\left(1+\sqrt{2}\right).
\end{aligned}$$

b) Suppose $\mathbf{D}$ has diagonal entries $d_1, \dots, d_n$, and assume w.l.o.g. that $|d_1| \geq |d_i|$ for $2 \leq i \leq n$.

For $\mathbf{e}_1 = (1, 0, \dots, 0)^\mathsf{T}$ we have $\mathbf{D}\mathbf{e}_1 = (d_1, 0, \dots, 0)^\mathsf{T}$ and hence $\frac{|\mathbf{D}\mathbf{e}_1|}{|\mathbf{e}_1|} = \frac{|d_1|}{1} = |d_1|$. This shows $\|\mathbf{D}\| \geq |d_1|$.

Now consider an arbitrary nonzero (column) vector $\mathbf{x} = (x_1, \dots, x_n)^\mathsf{T}$ in $\mathbb{R}^n$. Then

$$\begin{aligned}
\mathbf{D}\mathbf{x} &= (d_1 x_1, \dots, d_n x_n)^\mathsf{T}, \\
|\mathbf{D}\mathbf{x}|^2 &= d_1^2 x_1^2 + \cdots + d_n^2 x_n^2 \leq d_1^2(x_1^2 + \cdots + x_n^2) = d_1^2\,|\mathbf{x}|^2
\end{aligned}$$

$\implies |\mathbf{D}\mathbf{x}|\,/\,|\mathbf{x}| \leq |d_1|$. This shows $\|\mathbf{D}\| \leq |d_1|$, i.e., in all $\|\mathbf{D}\| = |d_1|$.

c) Since T is of class C^1, the map $\mathbf{x} \to \mathbf{J}_T(\mathbf{x})$ is continuous. (Its coordinate functions are the partial derivatives $\partial T_i/\partial x_j$.) Since $\mathbf{A} \mapsto \|\mathbf{A}\|$ is continuous as well (the triangle inequality implies $|\|\mathbf{A}\| - \|\mathbf{B}\|| \leq \|\mathbf{A} - \mathbf{B}\|$, so that $\delta = \epsilon$ works in a continuity proof), the composition $\mathbf{x} \mapsto \|\mathbf{J}_T(\mathbf{x})\|$ is continuous. Hence, since $\|\mathbf{J}_T(\mathbf{x}^*)\| = \|\mathbf{0}\| = 0$, there exists a ball $\mathrm{B}_r(\mathbf{x}^*)$, $r > 0$, such that $\|\mathbf{J}_T(\mathbf{x})\| < 1/2$ for $\mathbf{x} \in \mathrm{B}_r(\mathbf{x}^*)$. For

$\mathbf{x} \in \overline{\mathrm{B}_r(\mathbf{x}^*)}$ we then have $\|\mathbf{J}_T(\mathbf{x})\| \leq 1/2$. (The constant 1/2 is arbitrary; we could achieve $\|\mathbf{J}_T(\mathbf{x})\| < \epsilon$, for any given $\epsilon > 0$, by choosing r suitably.)

As shown in the lecture, we have $T(\mathbf{x}) - T(\mathbf{y}) = \mathbf{A}(\mathbf{x}-\mathbf{y})$ with $\mathbf{A} = \int_0^1 \mathbf{J}_T\big(\mathbf{x} + t(\mathbf{y}-\mathbf{x})\big)\,\mathrm{d}t$. For $\mathbf{x},\mathbf{y} \in \overline{\mathrm{B}_r(\mathbf{x}^*)}$ and $t \in [0,1]$ we also have $\mathbf{x} + t(\mathbf{y}-\mathbf{x}) \in \overline{\mathrm{B}_r(\mathbf{x}^*)}$ and hence $\big\|\mathbf{J}_T\big(\mathbf{x}+t(\mathbf{y}-\mathbf{x})\big)\big\| \leq 1/2$. Since for (continuous) matrix-valued functions $[a,b] \to \mathbb{R}^{n\times n}$, $t \mapsto \mathbf{M}(t)$ the inequality $\left\|\int_a^b \mathbf{M}(t)\,\mathrm{d}t\right\| \leq \int_a^b \|\mathbf{M}(t)\|\,\mathrm{d}t$ holds (if you don't believe this, use the corresponding inequality for the Frobenius norm instead and adapt the constants suitably), we obtain $\|\mathbf{A}\| \leq \int_0^1 1/2\,\mathrm{d}t = 1/2$ and $|T(\mathbf{x}) - T(\mathbf{y})| \leq \frac{1}{2}|\mathbf{x}-\mathbf{y}|$ for $\mathbf{x},\mathbf{y} \in \overline{\mathrm{B}_r(\mathbf{x}^*)}$.

d) We can take $\mathbf{A} = \begin{pmatrix} 3/4 & 0 \\ 0 & 3/4 \end{pmatrix}$. Since $\|\mathbf{A}\| = 3/4$, we have

$$d(\mathbf{A}\mathbf{x},\mathbf{A}\mathbf{y}) = |\mathbf{A}\mathbf{x} - \mathbf{A}\mathbf{y}| = |\mathbf{A}(\mathbf{x}-\mathbf{y})| \leq \|\mathbf{A}\|\,|\mathbf{x}-\mathbf{y}| = \tfrac{3}{4}|\mathbf{x}-\mathbf{y}| = \tfrac{3}{4}\,d(\mathbf{x},\mathbf{y}),$$

so that $\mathbf{x} \mapsto \mathbf{A}\mathbf{x}$ is a contraction. But $\|\mathbf{A}\|_{\mathrm{F}} = \sqrt{(3/4)^2 + (3/4)^2} = \sqrt{18/16} > 1$.

32 Note that solutions of all three ODE's must have non-negative derivative and hence cannot decrease anywhere strictly.

a) The function $f(t,y) = |y|$ is continuous and trivially satisfies a Lipschitz condition with respect to y (with Lipschitz constant $L = 1$, since $|f(t,y_1) - f(t,y_2)| = |y_1 - y_2| \leq 1 \cdot |y_1 - y_2|$. Hence solutions exist and are unique everywhere. The general solution is

$$y_C(t) = \begin{cases} C\,\mathrm{e}^t & \text{if } C \geq 0, \\ C\,\mathrm{e}^{-t} & \text{if } C < 0, \end{cases}$$

where C can be any real number. This follows by considering the three cases $y > 0$, $y = 0$, $y < 0$ separately.

b) $f(t,y) = \sqrt{|y - y^2|}$ is C^1 on the three plane regions $y < 0$, $0 < y < 1$, $y > 1$, and does not satisfy a Lipschitz condition with respect to y locally at points of the separating lines $y = 0$ and $y = 1$. The latter follows from the fact that the derivative $\frac{\partial f}{\partial y}$ is unbounded near $y = 0$ and $y = 1$. For example, for $0 < y < 1$ we have

$$|f(t,y) - f(t,1)| = \left|\frac{\partial f}{\partial y}(t,\eta)\right| |y - 1| = \left|\frac{1-2\eta}{2\sqrt{\eta - \eta^2}}\right| |y-1|$$

for some $\eta \in (y,1)$, and for y (and hence η) close to 1 the factor $\left|\frac{1-2\eta}{2\sqrt{\eta-\eta^2}}\right|$ becomes arbitarily large.

The Existence and Uniqueness Theorem gives that solutions exist and are unique locally at points within the three regions. At points (t,y) with $y \in \{0,1\}$ solutions are not unique as the following explicit solution shows.

$0 < y < 1$**:** $\mathrm{d}y/\sqrt{y-y^2} = 2\,\mathrm{d}y\,/\sqrt{1-(2y-1)^2} = 1 \Longrightarrow \arcsin(2y-1) = t + C \Longrightarrow$
$y = \frac{1}{2}\big(1 + \sin(t+C)\big) = \frac{1}{2}\big(1 + \cos(t+C')\big)$

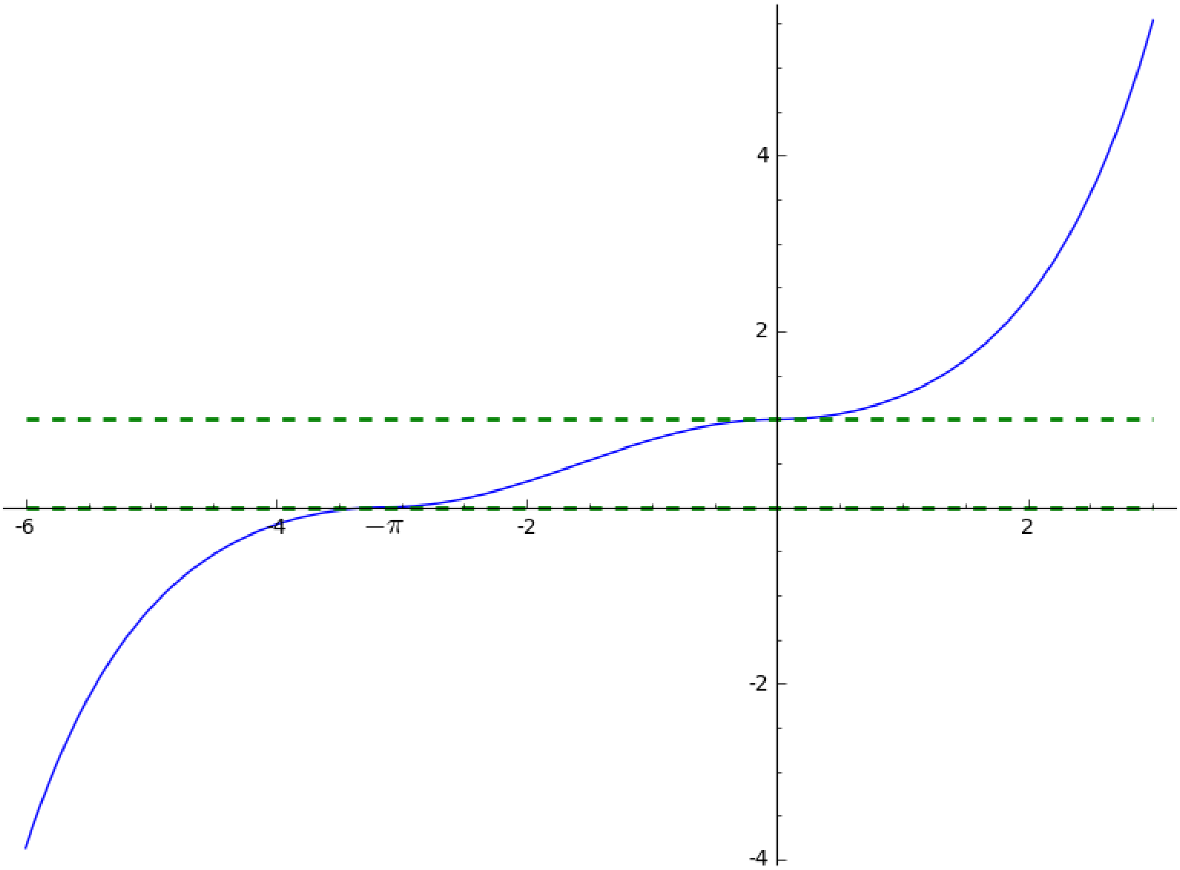


Figure 1: The solution $y(t)$ from H32b)

$$y > 1\colon\ \mathrm{d}y/\sqrt{y^2 - y} = 2\,\mathrm{d}y\,/\sqrt{(2y-1)^2 - 1} = 1 \Longrightarrow \operatorname{arcosh}(2y-1) = t + C \Longrightarrow y = \tfrac{1}{2}\big(1 + \cosh(t+C)\big)$$

$$y < 0\colon\ \mathrm{d}y/\sqrt{y^2 - y} = 2\,\mathrm{d}y\,/\sqrt{(1-2y)^2 - 1} = 1 \Longrightarrow -\operatorname{arcosh}(1-2y) = t + C \Longrightarrow y = \tfrac{1}{2}\big(1 - \cosh(-t+C')\big)$$

Solutions from the 3 cases can be glued together at $y = 0$ and $y = 1$ to satisfy the same initial conditions as the constant solutions. One particular example is

$$y(t) = \begin{cases} \frac{1}{2}(1 - \cosh(-t-\pi) & \text{for } t \leq -\pi, \\ \frac{1}{2}(1 + \cos t) & \text{for } -\pi \leq t \leq 0, \\ \frac{1}{2}(1 + \cosh t) & \text{for } t \geq 0; \end{cases}$$

see Figure 1. When constructing solutions, there is more degree of freedom, e.g., we can make solutions follow the line $y = 0$ for a while, then branch off and flow into the line $y = 1$, follow this line for another while, etc.

33 According to Picard-Lindelöf iteration, we have

$$\phi_{k+1}(t) = y_0 + \int_0^t f(s, \phi_k(s))\,ds, \quad k = 0, 1, 2, \ldots$$

Note that in this case $\phi_k(t)$ and y_0 are vectors in $\mathbb{R}^2$, and the notation used is somewhat inconsistent with "$\phi = (\phi_1, \phi_2)^\mathsf{T}$" in the statement of the exercise (but preferred for its simplicity).
Since $\phi_0(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = y_0$, we have

$$\begin{aligned}
\phi_1(t) &= y_0 + \int_0^t f(s, \phi_0(s))\, ds = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \int_0^t \begin{pmatrix} 0 \\ 1 \end{pmatrix} ds = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ t \end{pmatrix} = \begin{pmatrix} 1 \\ t \end{pmatrix}, \\
\phi_2(t) &= y_0 + \int_0^t f(s, \phi_1(s))\, ds = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \int_0^t \begin{pmatrix} -s \\ 1 \end{pmatrix} ds = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} \frac{t^2}{2} \\ t \end{pmatrix} = \begin{pmatrix} 1 - \frac{t^2}{2} \\ t \end{pmatrix}, \\
\phi_3(t) &= y_0 + \int_0^t f(s, \phi_2(s))\, ds = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \int_0^t \begin{pmatrix} -s \\ 1 - \frac{s^2}{2} \end{pmatrix} ds = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} -\frac{t^2}{2} \\ t - \frac{t^3}{6} \end{pmatrix} = \begin{pmatrix} 1 - \frac{t^2}{2} \\ t - \frac{t^3}{6} \end{pmatrix}, \\
\phi_4(t) &= y_0 + \int_0^t f(s, \phi_3(s))\, ds = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \int_0^t \begin{pmatrix} \frac{s^3}{6} - s \\ 1 - \frac{s^2}{2} \end{pmatrix} ds = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} \frac{t^4}{24} - \frac{t^2}{2} \\ t - \frac{t^3}{6} \end{pmatrix} = \begin{pmatrix} 1 - \frac{t^2}{2} + \frac{t^4}{24} \\ t - \frac{t^3}{6} \end{pmatrix}, \\
&\vdots \\
\phi_{2k-1}(t) &= \begin{pmatrix} 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \cdots + (-1)^{k-1} \frac{t^{2k-2}}{(2k-2)!} \\ t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \cdots + (-1)^{k-1} \frac{t^{2k-1}}{(2k-1)!} \end{pmatrix}, \\
\phi_{2k}(t) &= \begin{pmatrix} 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \cdots + (-1)^{k} \frac{t^{2k}}{2k!} \\ t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \cdots + (-1)^{k-1} \frac{t^{2k-1}}{(2k-1)!} \end{pmatrix}.
\end{aligned}$$

$$\implies \phi(t) = \begin{pmatrix} \sum_{k=0}^\infty (-1)^k \frac{t^{2k}}{2k!} \\ \sum_{k=0}^\infty (-1)^k \frac{t^{2k+1}}{(2k+1)!} \end{pmatrix} = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$$

34 Consider the function $\psi(t) = \phi(-t)$, also defined for $t \in [-r, r]$. We have $\psi(0) = \phi(0) = y_0$, say, and

$$\psi'(t) = -\phi'(-t) = -f\bigl(-t, \phi(-t)\bigr) = f\bigl(t, \phi(-t)\bigr) = f\bigl(t, \psi(t)\bigr).$$

Hence both ϕ and ψ solve the IVP $y' = f(t, y) \wedge y(0) = y_0$. Since f satisfies the assumptions in the Existence and Uniqueness Theorem(s), it follows that $\phi = \psi$, i.e., $\phi(t) = \phi(-t)$ for $t \in [-r, r]$. This is the indicated symmetry property.
Remark: It is sufficient to assume that f satisfies locally a Lipschitz condition with respect to y, which is weaker than "Lipschitz condition per se".

35 a) (N1), (N2) follow from the corresponding properties of the Euclidean length. For (N3) this is also true, but here we give a detailed proof: The triangle inequality for $|\cdot|$ yields for $\mathbf{x} \in \mathbb{R}^n$ with $|\mathbf{x}| = 1$ the estimate

$$|(\mathbf{A} + \mathbf{B})\mathbf{x}| = |\mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{x}| \le |\mathbf{A}\mathbf{x}| + |\mathbf{B}\mathbf{x}| \le \|\mathbf{A}\| + \|\mathbf{B}\|.$$

Taking the maximum over all such vectors $\mathbf{x}$ then gives

$$\|\mathbf{A} + \mathbf{B}\| = \max\bigl\{|(\mathbf{A} + \mathbf{B})\mathbf{x}|\,; \mathbf{x} \in \mathbb{R}^n, |\mathbf{x}| = 1\bigr\} \le \|\mathbf{A}\| + \|\mathbf{B}\|.$$

For (N4) we can argue as follows:

$$|(\mathbf{AB})\mathbf{x}| = |\mathbf{A}(\mathbf{Bx})| \leq \|\mathbf{A}\|\,|\mathbf{Bx}| \leq \|\mathbf{A}\|\,\|\mathbf{B}\|\,|\mathbf{x}| \implies \frac{|(\mathbf{AB})\mathbf{x}|}{|\mathbf{x}|} \leq \|\mathbf{A}\|\,\|\mathbf{B}\| \text{ for } \mathbf{x} \neq \mathbf{0}$$

Taking the maximum over all vectors $\mathbf{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ then gives $\|\mathbf{AB}\| \leq \|\mathbf{A}\|\,\|\mathbf{B}\|$. (Alternatively we can restrict the above computation to vectors $\mathbf{x}$ of length 1, resulting in $|(\mathbf{AB})\mathbf{x}| \leq \|\mathbf{A}\|\,\|\mathbf{B}\|$, and then take the maximum over those vectors as in the proof of (N3).)

b) Since the Frobenius norm is a matrix analogue of the Euclidean length on $\mathbb{R}^{n^2}$, it clearly satisfies (N1)–(N3). For the proof of (N4) we write $\mathbf{A} = (a_{ij})$, $\mathbf{B} = (b_{ij})$, so that $\mathbf{AB} = (c_{ij}) = \left(\sum_{k=1}^n a_{ik}b_{kj}\right)_{i,j=1}^n$. Denoting the i-th row of $\mathbf{A}$ by $\mathbf{a}_i$ and the j-th column of $\mathbf{B}$ by $\mathbf{b}_j$, we have

$$\begin{aligned} c_{ij} &= \mathbf{a}_i \cdot \mathbf{b}_j, \\ c_{ij}^2 &\leq |\mathbf{a}_i|^2\,|\mathbf{b}_j|^2. && \text{(Cauchy-Schwarz Inequality)} \end{aligned}$$

Summing these inequalities over i, j gives

$$\|\mathbf{AB}\|_{\mathrm{F}}^2 \leq \sum_{i,j=1}^n |\mathbf{a}_i|^2\,|\mathbf{b}_j|^2 = \left(\sum_{i=1}^n |\mathbf{a}_i|^2\right)\left(\sum_{j=1}^n |\mathbf{b}_j|^2\right) = \|\mathbf{A}\|_{\mathrm{F}}^2\,\|\mathbf{B}\|_{\mathrm{F}}^2,$$

and (N4) follows.

c) It suffices to show $|\mathbf{Ax}| \leq \|\mathbf{A}\|_{\mathrm{F}}$ for all vectors $\mathbf{x} \in \mathbb{R}^n$ with $|\mathbf{x}| = 1$. Using the notation introduced in b) we have

$$\begin{aligned} \mathbf{Ax} &= \begin{pmatrix} \mathbf{a}_1 \cdot \mathbf{x} \\ \vdots \\ \mathbf{a}_n \cdot \mathbf{x} \end{pmatrix}, \\ (\mathbf{a}_i \cdot \mathbf{x})^2 &\leq |\mathbf{a}_i|^2\,|\mathbf{x}|^2 = |\mathbf{a}_i|^2. \\ \implies |\mathbf{Ax}|^2 &= \sum_{i=1}^n (\mathbf{a}_i \cdot \mathbf{x})^2 \leq \sum_{i=1}^n |\mathbf{a}_i|^2 = \|\mathbf{A}\|_{\mathrm{F}}^2 \end{aligned}$$

This proves $|\mathbf{Ax}| \leq \|\mathbf{A}\|_{\mathrm{F}}$ and implies the desired inequality $\|\mathbf{A}\| \leq \|\mathbf{A}\|_{\mathrm{F}}$ for $\mathbf{A} \in \mathbb{R}^{n\times n}$. *Remark:* The matrix norms considered so far and their properties remain true if $\mathbb{R}^n$ is replaced by $\mathbb{C}^n$ and the Euclidean length on $\mathbb{R}^n$ by $|\mathbf{x}| = \sqrt{\sum_{i=1}^n |x_i|^2}$. The above proofs remain valid for $\mathbb{C}^n$, provided we change squares of real numbers to squared absolute values of complex numbers, e.g., $(\mathbf{a}_i \cdot \mathbf{x})^2$ becomes $|\mathbf{a}_i \cdot \mathbf{x}|^2$.

d) As shown at the end of `lecture14-16_handout.pdf`, $\|\mathbf{A}\|^2$ is the largest eigenvalue of $\mathbf{A}^\mathsf{T}\mathbf{A}$. Applying this to $\mathbf{A}^\mathsf{T}$, we see that $\|\mathbf{A}^\mathsf{T}\|^2$ is the largest eigenvalue of $\mathbf{A}\mathbf{A}^\mathsf{T}$. But the eigenvalues of $\mathbf{A}^\mathsf{T}\mathbf{A}$ and $\mathbf{A}\mathbf{A}^\mathsf{T}$ are the same (cf. Math257 in Fall 2022, Exercise H46 c) of Homework 9), and hence the same is true of the (spectral) norms of $\mathbf{A}$ and $\mathbf{A}^\mathsf{T}$.

e) If $\mathbf{A}$ is invertible then so is $\mathbf{A}^\top\mathbf{A}$ (cf. Math257 in Fall 2022, Exercise W26 a) of Worksheet 7). Hence hence all its eigenvalues are positive and $1/\sqrt{\lambda_n}$ is well-defined.

Since $\left(\mathbf{A}^{-1}\right)^\top \mathbf{A}^{-1} = \left(\mathbf{A}^\top\right)^{-1}\mathbf{A}^{-1} = \left(\mathbf{A}\mathbf{A}^\top\right)^{-1}$, the eigenvalues of $\left(\mathbf{A}^{-1}\right)^\top \mathbf{A}^{-1}$ are $\lambda_n^{-1} > \lambda_{n-1}^{-1} > \cdots > \lambda_1^{-1}$ (cf. Math257 in Fall 2022, Exercise W34 a) of Worksheet 10). Thus we have $\left\|\mathbf{A}^{-1}\right\|^2 = \lambda_n^{-1}$, as claimed.

f) The entries of $\mathbf{A}^\top\mathbf{A}$ are the pairwise dot products of the columns $\mathbf{c}_1, \ldots, \mathbf{c}_n$ of $\mathbf{A}$. In particular we have

$$\operatorname{tr}\left(\mathbf{A}^\top\mathbf{A}\right) = \sum_{i=1}^{n}\left(\mathbf{A}^\top\mathbf{A}\right)_{ii} = |\mathbf{c}_1|^2 + \cdots + |\mathbf{c}_n|^2 = \sum_{i,j=1}^{n} a_{ij}^2 = \|\mathbf{A}\|_{\mathrm{F}}^2 .$$

On the other hand, $\operatorname{tr}\left(\mathbf{A}^\top\mathbf{A}\right)$ is equal to the sum of the eigenvalues of $\mathbf{A}^\top\mathbf{A}$. Both identities taken together give $\|\mathbf{A}\|_{\mathrm{F}}^2 = \lambda_1 + \cdots + \lambda_n$, i.e., $\|\mathbf{A}\|_{\mathrm{F}} = \sqrt{\lambda_1 + \cdots + \lambda_n}$.

Differential Equations (Math 285)

H36 Use the phase line to investigate the stability of the equilibrium solutions of the following autonomous ODE's.

a) $y' = 2(1-y)(1-\mathrm{e}^y)$; b) $y' = (1-y^2)(4-y^2)$; c) $y' = \sin^2 y$.

H37 *From a previous final exam*

Consider the differential equation

$$(3xy + 2y^2)\,\mathrm{d}x + (3x^2 + 6xy + 3y^2)\,\mathrm{d}y = 0. \tag{DF}$$

a) Show that $(0,0)$ is the only singular point of (DF).

b) Transform (DF) into an exact equation and determine the general solution in implicit form.

c) Is every point of $\mathbb{R}^2$ on a unique integral curve of (DF)?

H38 Determine a real fundamental system of solutions for the following ODE's:

a) $y'' - 4y' + 4y = 0$;

b) $y''' - 2y'' - 5y' + 6y = 0$;

c) $y''' - 2y'' + 2y' - y = 0$;

d) $y''' - y = 0$;

e) $y^{(4)} + y = 0$;

f) $y^{(8)} + 4y^{(6)} + 6y^{(4)} + 4y'' + y = 0$.

Four answers suffice.

H39 Determine the general real solution of

a) $y'' + 3y' + 2y = 2$;

b) $y'' + y' - 12y = 1 + t^2$;

c) $y'' - 5y' + 6y = 4t\mathrm{e}^t - \sin t$;

d) $y''' - 2y'' + y' = 1 + \mathrm{e}^t\cos(2t)$;

e) $y^{(4)} + 2y'' + y = 25\mathrm{e}^{2t}$;

f) $y^{(n)} = t\mathrm{e}^t$, $n \in \mathbb{N}$.

Four answers suffice.

Due on Thu Apr 11, 10 am

General solution techniques for higher-order linear ODE's with constant coefficients (required for H38 and H39) will be discussed in the lecture on Wed April 3.

Solutions (prepared by Liang Tingou and TH)

36 a) Setting $y' = 2(1-y)(1-e^y) = 0$ gives the two equilibrium solutions $y_1 = 0$, $y_2 = 1$. The graph of y' versus y is shown below. So $y_1 = 0$ is an asymptotically stable stable

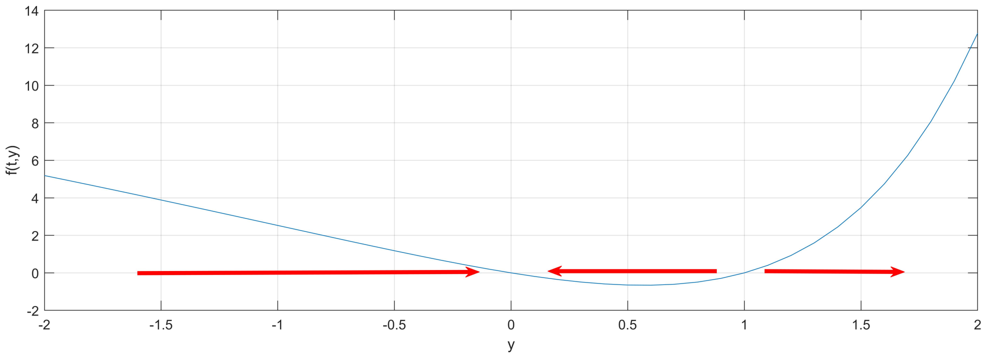


Figure 1: H36 a)

equilibrium, while $y_2 = 1$ is an unstable equilibrium.

b) Setting $y' = (1-y^2)(4-y^2) = 0$ gives the four equilibria $y_1 = -2, y_2 = -1, y_3 = 1, y_4 = 2$. The graph of y' versus y is shown below. So $y_1 = -2$, $y_3 = 1$ are asymptotically

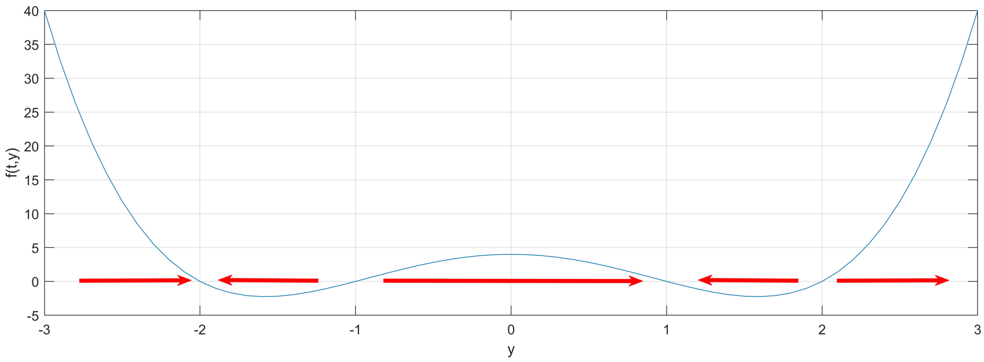


Figure 2: H36 b)

stable solutions, while $y_2 = -1, y_4 = 2$ are unstable solutions.

c) Setting $y' = \sin^2 y = 0$ gives infinitely many equilibrium solutions, viz. $y_k = k\pi$ ($k \in \mathbb{Z}$). The graph of y' versus y is shown below. So all equilibria are semistable (asymptotically stable from below, unstable from above).

37 a) $M(x,y) = 3xy + 2y^2 = y(3x+2y)$, $N(x,y) = 3x^2 + 6xy + 3y^2 = 3(x+y)^2$ have no common zero except $(0,0)$. $\Longrightarrow$ $(0,0)$ is the only singular point.

b) We have

$$M_y - N_x = 3x + 4y - (6x + 6y) = -3x - 2y = M(-1/y).$$

Thus $(M_y - N_x)/M$ depends only on y, and there is an integrating factor of the form $g(y)$.

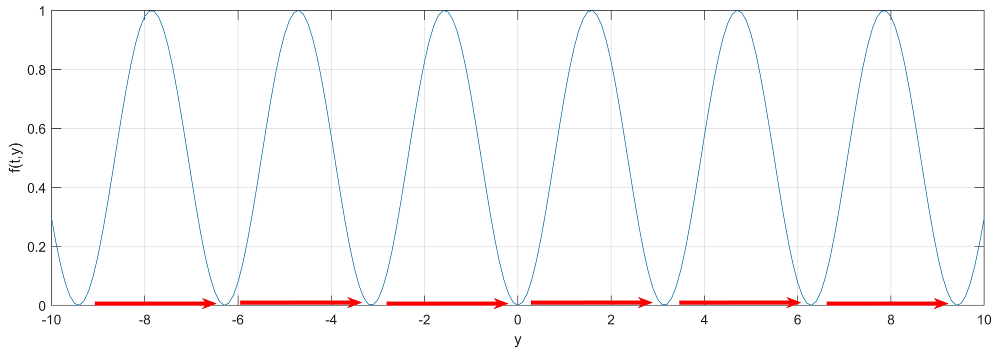


Figure 3: H36 c)

The integrability condition $(gM)_y = (gN)_x$ then becomes $g'M + gM_y = gN_x$, i.e.,

$$g' = \frac{g(N_x - M_y)}{M} = \frac{g}{y}.$$

The solution of this ODE is $g(y) = cy$, so that we can take $g(y) = y$.
$\Longrightarrow$ On $\mathbb{R}^2 \setminus x$-axis the ODE $(3xy + 2y^2)\,\mathrm{d}x + (3x^2 + 6xy + 3y^2)\,\mathrm{d}y = 0$ is equivalent to the exact ODE

$$\big(3xy^2 + 2y^3\big)\,\mathrm{d}x + \big(3x^2y + 6xy^2 + 3y^3\big)\,\mathrm{d}y = 0.$$

An antiderivative f of the corresponding exact differential is determined in the usual way by "partial integration" with respect to x, say.

$$\begin{aligned} f(x,y) &= \tfrac{3}{2}\,x^2y^2 + 2\,xy^3 + g(y), \\ f_y(x,y) &= 3\,x^2y + 6\,xy^2 + g'(y) \overset{!}{=} 3x^2y + 6xy^2 + 3y^3 \end{aligned}$$

$\Longrightarrow g'(y) = 3\,y^3 \Longrightarrow g(y) = \frac{3}{4}\,y^4 + C \Longrightarrow f(x,y) = \frac{3}{2}\,x^2y^2 + 2\,xy^3 + \frac{3}{4}\,y^4 + C$
The general implicit solution of the exact ODE is then given by (in slightly simplified form and with a different C)

$$6\,x^2y^2 + 8\,xy^3 + 3\,y^4 = C, \quad C \in \mathbb{R}.$$

Solutions with $C < 0$ don't exist and for $C = 0$ the x-axis is obtained, since $6\,x^2y^2 + 8\,xy^3 + 3\,y^4 = y^2(6x^2 + 8xy + 3y^2)$ and the quadratic has discriminant $8^2 - 4 \cdot 6 \cdot 3 = -8 < 0$.

Since the x-axis (equivalently, the function $y(x) \equiv 0$) is a solution of (DF), multiplication by y hasn't introduced any new solution, and $6\,x^2y^2 + 8\,xy^3 + 3\,y^4 = C$, $C \geq 0$ solves (DF) as well.

c) Yes. This is implicit in the preceding discussion. Intersection points of integral curves must be singular, so that the only candidate for such a point is the origin. But the corresponding contour of f, the 0-contour, consists of a single integral curve, viz. the x-axis.

Remark: Part c) serves as an illustration for the fact that at singular points virtually anything can happen. Here it is due to the fact that at $(0,0)$ the partial derivatives of f up to order 3 vanish. In the lecture we have seen an example of a singular point being on exactly two integral curves (Example 8 of the introduction, `lecture1-3_handout.pdf, Slides 31 ff`), and another one with a singular point contained in infinitely many integral curves (the example at the end of `lecture17-18_handout.pdf`). Singular points contained in no integral curve are also possible: For exact equations $\omega = \mathrm{d}f = 0$ this happens at a strict local extremum of f, where the corresponding contour reduces to a single point (at least locally).

38 a) The characteristic polynomial is $a(X) = X^2 - 4X + 4$.
The only root is $x = 2$ with multiplicity 2.
So, a real fundamental system of solutions is e^{2t}, te^{2t}.

b) The characteristic polynomial is

$$\begin{aligned} a(X) &= X^3 - 2X^2 - 5X + 6 \\ &= (x-1)(x+2)(x-3) \end{aligned}$$

The roots are $x_1 = -2$, $x_2 = 1$, $x_3 = 3$, all with multiplicity 1.
So, a real fundamental system of solutions is e^{-2t}, e^t, e^{3t}.

c) The characteristic polynomial is

$$\begin{aligned} a(X) &= X^3 - 2X^2 + 2X - 1 \\ &= (X-1)\left(X - \frac{1-\sqrt{3}\,\mathrm{i}}{2}\right)\left(X - \frac{1+\sqrt{3}\,\mathrm{i}}{2}\right) \end{aligned}$$

The roots are $x_1 = 1$, $x_2 = \frac{1-\sqrt{3}\,\mathrm{i}}{2}$, $x_3 = \frac{1+\sqrt{3}\,\mathrm{i}}{2}$ with multiplicities 1.
So, a real fundamental system of solutions is e^t, $e^{\frac{t}{2}}\cos\left(\frac{\sqrt{3}}{2}t\right)$, $e^{\frac{t}{2}}\sin\left(\frac{\sqrt{3}}{2}t\right)$.

d) The characteristic polynomial is

$$\begin{aligned} a(X) &= X^3 - 1 \\ &= (X-1)\left(X - \frac{-1-\sqrt{3}\,\mathrm{i}}{2}\right)\left(X - \frac{-1+\sqrt{3}\,\mathrm{i}}{2}\right) \end{aligned}$$

The roots are 1, $\frac{-1-\sqrt{3}\,\mathrm{i}}{2}$, $\frac{-1+\sqrt{3}\,\mathrm{i}}{2}$ with multiplicities 1.
So, a real fundamental system of solutions is e^t, $e^{-t/2}\cos\left(\frac{\sqrt{3}}{2}t\right)$, $e^{-t/2}\sin\left(\frac{\sqrt{3}}{2}t\right)$.

e) The characteristic polynomial is

$$\begin{aligned} a(X) &= X^4 + 1 \\ &= X^4 + 2X^2 + 1 - 2X^2 \\ &= (X^2+1)^2 - (\sqrt{2}X)^2 \\ &= (X^2 + \sqrt{2}X + 1)(X^2 - \sqrt{2}X + 1) \\ &= \left(X - \frac{-\sqrt{2}-\sqrt{2}\,\mathrm{i}}{2}\right)\left(X - \frac{-\sqrt{2}+\sqrt{2}\,\mathrm{i}}{2}\right)\left(X - \frac{\sqrt{2}-\sqrt{2}\,\mathrm{i}}{2}\right)\left(X - \frac{\sqrt{2}+\sqrt{2}\,\mathrm{i}}{2}\right) \end{aligned}$$

The roots are $\frac{\pm\sqrt{2}\pm\sqrt{2}\,\mathrm{i}}{2}$ (all 4 combinations) with multiplicities 1.
So, a real fundamental system of solutions is $e^{-\frac{\sqrt{2}}{2}t}\cos\big(\frac{\sqrt{2}}{2}t\big)$, $e^{-\frac{\sqrt{2}}{2}t}\sin\big(\frac{\sqrt{2}}{2}t\big)$, $e^{\frac{\sqrt{2}}{2}t}\cos\big(\frac{\sqrt{2}}{2}t\big)$, $e^{\frac{\sqrt{2}}{2}t}\sin\big(\frac{\sqrt{2}}{2}t\big)$.

f) The characteristic polynomial is

$$\begin{aligned} a(X) &= X^8 + 4X^6 + 6X^4 + 4X^2 + 1 \\ &= (X^2+1)^4 \\ &= (X-\mathrm{i})^4(X+\mathrm{i})^4 \end{aligned}$$

The roots are $x_1 = \mathrm{i}$ and $x_2 = -\mathrm{i}$, each with multiplicity 4.
So, a real fundamental system of solutions is
$\cos(t)$, $t\cos(t)$, $t^2\cos(t)$, $t^3\cos(t)$, $\sin(t)$, $t\sin(t)$, $t^2\sin(t)$, $t^3\sin(t)$.

39 a) Using the method for determining the solution of inhomogeneous linear ODE's stated in the lecture slides, we have $b(t) = 2 = 2e^{0t}$.
So, $\mu = 0$.
The characteristic polynomial is

$$\begin{aligned} a(X) &= X^2 + 3X + 2 \\ &= (X+1)(X+2) \end{aligned}$$

The roots are $x_1 = -1$ and $x_2 = -2$ with multiplicities 1.
So, e^{-t}, e^{-2t} form a fundamental system of solutions.
Since $\mu = 0$ has multiplicity 0, there exists a particular solution of the form $y_p(t) = c_0$.
Substituting it into the ODE gives

$$2c_0 = 2,$$

which gives $c_0 = 1$. (Alternatively, $y_p(t) = \frac{1}{a(\mu)}\, b(t) = \frac{2}{a(0)} = 1$.)
So, the general real solution is $y(t) = c_1 e^{-t} + c_2 e^{-2t} + 1$, $c_i \in \mathbb{R}$.

b) $b(t) = 1 + t^2 = (1+t^2)e^{0t}$
So, $\mu = 0$.
The characteristic polynomial is

$$\begin{aligned} a(X) &= X^2 + X - 12 \\ &= (X+4)(X-3) \end{aligned}$$

The roots are $x_1 = -4$ and $x_2 = 3$ with multiplicity 1.
So, e^{-4t}, e^{3t} form a fundamental system of solutions.
Since $\mu = 0$ has multiplicity 0, there exists a particular solution of the form $y(t) = c_0 + c_1 t + c_2 t^2$. Substituting it into the ODE gives

$$\begin{aligned} 2c_2 + 2c_2 t + c_1 - 12(c_2 t^2 + c_1 t + c_0) &= 1 + t^2, \\ -12c_2 t^2 + (-12c_1 + 2c_2)t + (-12c_0 + c_1 + 2c_2) &= 1 + t^2. \end{aligned}$$

The solution is

$$\begin{cases} c_2 = -\frac{1}{12} \\ c_1 = -\frac{1}{72} \\ c_0 = -\frac{85}{864} \end{cases}$$

Therefore, the general real solution is $y(t) = c_1 e^{-4t} + c_2 e^{3t} - \frac{85}{864} - \frac{1}{72}t - \frac{1}{12}t^2$, $c_i \in \mathbb{R}$.

c) The characteristic polynomial is

$$\begin{aligned} a(X) &= X^2 - 5X + 6 \\ &= (X-2)(X-3) \end{aligned}$$

The roots are $x_1 = 2$ and $x_2 = 3$ with multiplicity 1. So, e^{2t}, e^{3t} form a fundamental system of solutions.
We now calculate particular solutions for $y'' - 5y' + 6y = 4te^t$ and $y'' - 5y' + 6y = -\sin(t)$.

i) $y'' - 5y' + 6y = 4te^t$
$\mu = 1$ has multiplicity 0.
So, the correct „Ansatz“ is $y_1(t) = (c_0 + c_1 t)e^t$, $c_i \in \mathbb{R}$. Substituting it into the ODE gives

$$(2c_0 - 3c_1 + 2c_1 t)e^t = 4te^t,$$

which gives

$$\begin{cases} c_1 = 2 \\ c_0 = 3 \end{cases}$$

So, $y_1(t) = (3 + 2t)e^t$.

ii) $y'' - 5y' + 6y = -\sin(t)$
We consider the "complexified" ODE $y'' - 5y' + 6y = -e^{\mathrm{i}t}$. The imaginary part of any particular solution of the complex ODE will solve the real ODE.
$b(t) = -e^{\mathrm{i}t}$ gives $\mu = i$, which has multiplicity 0.
So, the complex ODE has a particular solution of the form $y_c(t) = a_0 e^{\mathrm{i}t}$, $a_0 \in \mathbb{C}$. Substituting it into the ODE gives

$$5a_0 e^{\mathrm{i}t} - 6\mathrm{i}e^{\mathrm{i}t} = -e^{\mathrm{i}t},$$

which gives $a_0 = -\frac{1}{10} - \frac{1}{10}\mathrm{i}$ and $y_c(t) = \frac{-1-\mathrm{i}}{10}\,\mathrm{e}^{\mathrm{i}t}$.
(Alternatively, $y_c(t) = \frac{-1}{a(\mathrm{i})}\,\mathrm{e}^{it} = \frac{-1}{\mathrm{i}^2 - 5\mathrm{i} + 6}\,\mathrm{e}^{it} = \frac{-1}{5-5\mathrm{i}}\,\mathrm{e}^{it} = \frac{-1-\mathrm{i}}{10}\,\mathrm{e}^{it}$.)
Therefore, $y_2(t) = -\frac{1}{10}\big(\sin(t) + \cos(t)\big)$

In all, a particular solution of $y'' - 5y' + 6y = 4\mathrm{t}e^t - \sin t$ is $y_p(t) = (3 + 2t)e^t - \frac{1}{10}(\sin(t) + \cos(t))$. Therefore, the general real solution of this ODE is

$$y(t) = c_1 e^{2t} + c_2 e^{3t} + 3e^t + 2\mathrm{t}e^t - \tfrac{1}{10}\sin(t) - \tfrac{1}{10}\cos(t), \quad c_i \in \mathbb{R}.$$

d) The characteristic polynomial is

$$\begin{aligned} a(X) &= X^3 - 2X^2 + X \\ &= X(X-1)^2. \end{aligned}$$

The roots are $x_1 = 0$ with multiplicity 1 and $x_2 = 1$ with multiplicity 2. So, 1, e^t, te^t form a fundamental system of solutions.
We now calculate particular solutions for $y''' - 2y'' + y' = 1$ and $y''' - 2y'' + y' = e^t \cos(2t)$.

i) $y''' - 2y'' + y' = 1$
$\mu = 0$ has multiplicity 1. So, $y_1(t) = c_0 t$. Substituting it into the ODE gives $c_0 = 1$, which gives $y_1(t) = t$. (This solution can also be found by just looking at the ODE.)

ii) $y''' - 2y'' + y' = e^t \cos(2t)$
We consider the "complexified" ODE $y''' - 2y'' + y' = e^{(1+2\mathrm{i})t}$. The real part of any particular solution of the complex ODE will solve the real ODE.
$b(t) = e^{(1+2\mathrm{i})t}$ gives $\mu = 1 + 2\mathrm{i}$, which has multiplicity 0.
So, the complex ODE has a solution of the form $y_c(t) = a_0 e^{(1+2\mathrm{i})t}$, $a_0 \in \mathbb{C}$. Substituting it into the ODE gives

$$a_0[(1+2\mathrm{i})^3 - 2(1+2\mathrm{i})^2 + (1+2\mathrm{i})]e^{(1+2\mathrm{i})t} = e^{(1+2\mathrm{i})t},$$

which gives $a_0 = -\frac{1}{20} + \frac{1}{10}\mathrm{i}$. (Alternatively, $a_0 = \frac{1}{a(1+2\mathrm{i})}$, which leads to the same result.)
So, $y_2(t) = -\frac{1}{20}\,\mathrm{e}^t \cos(2t)) - \frac{1}{10}\,\mathrm{e}^t \sin(2t)$.

Therefore, the general real solution of $y''' - 2y'' + y' = 1 + \mathrm{e}^t \cos(2t)$ is

$$y(t) = c_0 + c_1 \mathrm{e}^t + c_2 t\mathrm{e}^t + t - \tfrac{1}{20}\,\mathrm{e}^t \cos(2t)) - \tfrac{1}{10}\,\mathrm{e}^t \sin(2t), \quad c_i \in \mathbb{R}.$$

e) $b(t) = 25e^{2t}$ gives $\mu = 2$.
The characteristic polynomial is

$$\begin{aligned} a(X) &= X^4 + 2X^2 + 1 \\ &= (X^2+1)^2 \\ &= (X+i)^2(X-i)^2 \end{aligned}$$

The roots are $x_1 = \mathrm{i}$ and $x_2 = -\mathrm{i}$, both with multiplicity 2, which means $\mu = 2$ has multiplicity 0.
$\cos(t)$, $t\cos(t)$, $\sin(t)$, $t\sin(t)$ form a fundamental system of solutions.
There exists a particular solution of the form $y_p(t) = c_0 e^{2t}$. Substituting it into the ODE gives

$$c_0(2^4 + 2 \times 2^2 + 1)e^{2t} = 25e^{2t},$$

which gives $c_0 = 1$. (Alternatively, $c_0 = \frac{25}{a(2)} = \frac{25}{25} = 1$.)
Therefore, the general real solution is

$$y(t) = c_1 \cos(t) + c_2 t\cos(t) + c_3 \sin(t) + c_4 t \sin(t) + e^{2t}, \quad c_i \in \mathbb{R}.$$

f) $b(t) = te^t$ gives $\mu = 1$.
The characteristic polynomial is $a(X) = X^n$ with root 0 of multiplicity n. (For $n = 0$ there is no root, but the multiplicity is still correct.)
Then, 1, t, t^2, ..., t^{n-1} form a fundamental system of solutions.

A particular solution is given by $y(t) = (c_0 + c_1 t)e^t$.
Substituting it into the ODE, we get

$$(c_0 + nc_1 + c_1 t)e^t = te^t,$$

which gives $c_1 = 1$ and $c_0 = -n$.
So, the general real solution is

$$y(t) = (t - n)e^t + \sum_{i=0}^{n-1} a_i t^i, \quad a_i \in \mathbb{R}.$$

For $n = 0$ this is true as well.

Differential Equations (Math 285)

H40 a) Suppose $\phi\colon \mathbb{R} \to \mathbb{C}$ solves a homogeneous linear ODE

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0, \quad a_i \in \mathbb{C}, \tag{H}$$

but no such ODE of order $< n$. Show that ϕ, ϕ', ϕ'', ..., $\phi^{(n-1)}$ form a fundamental system of solutions of (H).

b) Find a fundamental system of solutions of the form ϕ, ϕ', ϕ'', ϕ''' for the ODE $y^{(4)} - y^{(3)} - y' + y = 0$.

H41 Do three of the four Exercises 4, 6, 14, 16 in the previous edition of our Calculus textbook [Ste16], Ch. 17.3.
You may need to study the relevant material in [Ste16], Ch. 17, or [BDM17], Ch. 3.7, 3.8 first.

H42 For $\alpha, \beta \in \mathbb{C}$ consider the explicit so-called Euler equation

$$y'' + \frac{\alpha}{t}\, y' + \frac{\beta}{t^2}\, y = 0 \qquad (t > 0). \tag{1}$$

a) Show that $\phi\colon \mathbb{R}^+ \to \mathbb{C}$ is a solution of (1) iff $\psi\colon \mathbb{R} \to \mathbb{C}$ defined by $\psi(s) = \phi(\mathrm{e}^s)$ is a solution of

$$y'' + (\alpha - 1)y' + \beta y = 0. \tag{2}$$

b) Using a), determine the general solution of (1) for $(\alpha, \beta) = (6, 4)$ and $(3, 1)$.

H43 *Optional Exercise*
Suppose that $y\colon \mathbb{R} \to \mathbb{C}$ solves some homogeneous linear ODE $a(D)y = y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0$ with coefficients $a_i \in \mathbb{C}$ (i.e., y is an exponential polynomial). Show:

a) There is a unique monic polynomial $m(X) \in \mathbb{C}[X]$ of smallest degree satisfying $m(\mathrm{D})y = 0$.

b) If $b(X) \in \mathbb{C}[X]$ satisfies $b(\mathrm{D})y = 0$ then $m(X)$ divides $b(X)$.

Hint: There is a link with the annihilator polynomials (periods) discussed in Math 257; see the section on companion matrices in `lecture19-24_handout.pdf`.

H44 *Optional Exercise*
For the following functions ϕ_i, find the homogeneous linear ODE $y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0$ ($a_i \in \mathbb{C}$) of smallest order having ϕ_i as a solution; cf. H43.

a) $\phi_1(t) = 2\sin t - 3\cos(3t)$;

b) $\phi_2(t) = \sin t \cos(3t)$;

c) $\phi_3(t) = -1 + t\mathrm{e}^{-2t}\cos t$;

d) $\phi_4(t) = \mathrm{e}^t + t^{1949} + t^{2019}$.

H45 *Optional Exercise*

This exercise shows that characteristic polynomials $a(X)$ of homogeneous linear ODEs $a(\mathrm{D})y = 0$, respectively, homogeneous linear recurrence relations $a(\mathrm{S})\mathbf{y} = \mathbf{0}$ are characteristic polynomials in the sense of Linear Algebra.

a) Show that the (complex) solution space V of $a(\mathrm{D})y = 0$ is D-invariant, and that the characteristic polynomial of the restriction $\mathrm{D}|_V$ is equal to $a(X)$.

b) Show that the (complex) solution space V of $a(\mathrm{S})\mathbf{y} = \mathbf{0}$ is S-invariant, and that the characteristic polynomial of the restriction $\mathrm{S}|_V$ is equal to $a(X)$.

Hint: In Math 257 we have characterized endomorphisms of finite-dimensional vector spaces whose minimum polynomial equals the characteristic polynomial; see the section on companion matrices in `lecture19-24_handout.pdf`. Do b) first, which is easier.

H46 *Optional Exercise*

In the lecture we have found that the ODE $y'' - y' - y = 1$ and its discrete "analogue" $y_{i+2} - y_{i+1} - y_i = 1$ both have the constant function $y(t) \equiv -1$ as a solution (of course, with different domains $\mathbb{R}$ resp. $\mathbb{N}$). Is this a pure coincidence or an instance of a more general correspondence between the continuous and discrete case?

Hint: It may help to identify the discrete analogue of the exponential function e^t first.

Due on Thu Apr 18, 10 am

The optional exercises may be handed in until Thu Apr 25, 10 am.

Solutions

40 a) A linear dependency relation $c_0\phi(t)+c_1\phi'(t)+c_2\phi''(t)+\cdots+c_{n-1}\phi^{(n-1)}(t)=0$, $t\in\mathbb{R}$, says that ϕ solves the ODE $(c_0/c_m)y+(c_1/c_m)y'+\cdots+(c_{m-1}/c_m)y^{(m-1)}+y^{(m)}=0$, where $m=\max\{0\leq i\leq n-1; c_i\neq 0\}$. Since this ODE has order smaller than n, under the given assumption this is impossible. Hence ϕ, ϕ', ϕ'', ..., $\phi^{(n-1)}$ are linearly independent in $\mathbb{C}^{\mathbb{R}}$.

Differentiating both sides of (H) shows that ϕ' is a solution of (H) as well. Repeating this argument then gives that all derivatives of ϕ are solutions. Since we know that the solution space S of (H) has dimension n, the n linearly independent solutions ϕ, ϕ', ϕ'', ..., $\phi^{(n-1)}$ must form a basis of S.

b) The characteristic polynomial of this ODE is $X^4-X^3-X+1=(X^3-1)(X-1)=(X^2+X+1)(X-1)^2$. Its roots are $\lambda_1=1$ of multiplicity 2, and $\lambda_{2/3}=-\frac{1}{2}\pm\frac{\sqrt{3}}{2}\mathrm{i}$ of multiplicity 1. A fundamental system of solutions is therefore e^t, $t\,\mathrm{e}^t$, $\mathrm{e}^{(-1+\sqrt{3}\,\mathrm{i})t/2}$, $\mathrm{e}^{(-1-\sqrt{3}\,\mathrm{i})t/2}$. We need to find a linear combination of these functions which doesn't solve an ODE of the same type and order <4. Using the known properties of polynomial differential operators $p(\mathrm{D})$ (cf. lecture) and H43b), it can be shown that any solution $\phi(t)=c_1\mathrm{e}^t+c_2t\,\mathrm{e}^t+c_3\mathrm{e}^{(-1+\sqrt{3}\,\mathrm{i})t/2}+c_4\mathrm{e}^{(-1-\sqrt{3}\,\mathrm{i})t/2}$ with $c_2c_3c_4\neq 0$ has this property. (Sketch of proof: $p(\mathrm{D})\phi(t)=c_1p(1)\mathrm{e}^t+c_2p(\mathrm{D})[t\mathrm{e}^t]+c_3p(\lambda_2)\mathrm{e}^{\lambda_2t}+c_4p(\lambda_3)\mathrm{e}^{\lambda_3t}=0$ iff $c_1p(1)\mathrm{e}^t+c_2p(\mathrm{D})[t\mathrm{e}^t]=c_3p(\lambda_2)\mathrm{e}^{\lambda_2t}=c_4p(\lambda_3)\mathrm{e}^{\lambda_3t}=0$ (since e^t, $t\mathrm{e}^t$, e^{λ_2t}, e^{λ_3t} are linearly independent and $p(\mathrm{D})[t\mathrm{e}^t]\in\langle\mathrm{e}^t,t\mathrm{e}^t\rangle$) iff $p(X)$ is divisible by $(X-1)^2$, $X-\lambda_2$, and $X-\lambda_3$, which in turn implies that $p(\mathrm{D})$ is divisible by $(X-1)^2(X-\lambda_2)(X-\lambda_3)=X^4-X^3-X+1$.) For example, we can take

$$\phi(t)=t\,\mathrm{e}^t+\frac{1}{2}\mathrm{e}^{(-1+\sqrt{3}\,\mathrm{i})t/2}+\frac{1}{2}\mathrm{e}^{(-1-\sqrt{3}\,\mathrm{i})t/2}=t\,\mathrm{e}^t+\mathrm{e}^{-t/2}\cos\left(\sqrt{3}\,t/2\right).$$

Remark: In Linear Algebra terms, a fundamental system of solutions of $a(\mathrm{D})y=0$ of the form $y,y',\dots,y^{(n-1)}$, $n=\deg a(X)$, amounts to a representation of the solution space as D-cyclic span $\langle y,\mathrm{D}y,\dots,\mathrm{D}^{n-1}y\rangle$. Such a generator y always exists; cf. the solution to H45 a). A different proof, using polynomial arithmetic, can be inferred from the solution above.

41 1) Exercise 4

a) From Hooke's Law, the force required to stretch the spring is

$$k(0.25)=13,$$

so $k=13/0.25=52$ [N/m]. Adopting the standard units of measurement (N for forces, kg for masses, seconds (s) for time, m for lengths), we get the ODE

$$2\,\frac{d^2x}{dt^2}+8\,\frac{dx}{dt}+52\,x=0$$

The characteristic polynomial of this ODE is $X^2+4X+26$, with roots $x_{1/2}=-2\pm\sqrt{22}\,\mathrm{i}$, and the solution is

$$x(t)=e^{-2t}\big(c_1\cos(\sqrt{22}\,t)+c_2\sin(\sqrt{22}\,t)\big).$$

Since $x(0) = 0$, we have $c_1 = 0$.

$$x'(t) = -2c_2 e^{-2t} \sin(\sqrt{22}\, t) + c_2 \sqrt{22}\, e^{-2t} \cos(\sqrt{22}\, t)$$

Since $x'(0) = 0.5$, we have $c_2 = \frac{1}{2\sqrt{22}}$. So, the position (measured in m) at time t (measured in s) is

$$x(t) = e^{-2t} \frac{1}{2\sqrt{22}} \sin(\sqrt{22}\, t).$$

b)

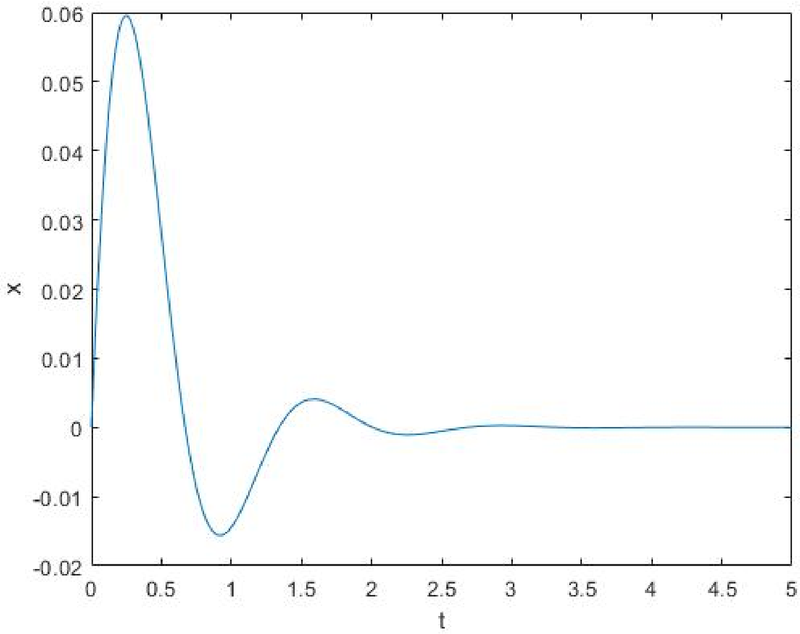


Figure 1: $x(t) = e^{-2t} \frac{1}{2\sqrt{22}} \sin(\sqrt{22}\, t)$

2) Exercise 6
The condition for critical damping is

$$c^2 = 4mk = 4 \times 2 \times 52 = 4^2 \times 26$$

So, $c = 4\sqrt{26}$ [N/m] will produce critical damping.

3) Exercise 14

a) By Kirchhoff's voltage law (and using the indicated standard units of measurement), we have

$$2\, \frac{d^2Q}{dt^2} + 24\, \frac{dQ}{dt} + 200\, Q = 12.$$

The (monic) characteristic polynomial of this ODE is

$$a(X) = X^2 + 12X + 100.$$

The roots are $X = -6 \pm 8\mathrm{i}$, so the solution of the associated homogeneous ODE (called *complementary equation* in [Ste16]) is

$$Q_c(t) = e^{-6t}\big(c_1 \cos(8t) + c_2 \sin(8t)\big).$$

For the method of undetermined coefficients we try a constant solution

$$\begin{aligned} Q_p(t) &= A, \\ Q_p'(t) = Q_p''(t) &= 0. \end{aligned}$$

Inserting this into the ODE gives $A = \frac{12}{200} = \frac{3}{50}$, so a particular solution is

$$Q_p(t) \equiv \frac{3}{50}$$

and the general solution is

$$\begin{aligned} Q(t) &= Q_c(t) + Q_p(t) \\ &= e^{-6t}\big(c_1 \cos(8t) + c_2 \sin(8t)\big) + \frac{3}{50}. \end{aligned}$$

The corresponding current is

$$\begin{aligned} I(t) &= \frac{dQ}{dt} \\ &= \mathrm{e}^{-6t}[(-6c_1 + 8c_2)\cos(8t) + (-8c_1 - 6c_2)\sin(8t)]. \end{aligned}$$

Imposing the initial conditions $Q(0) = 0.001$ and $I(0) = 0$, we get $c_1 = -0.059$, $c_2 = -0.04425$.
Thus, the formula for the charge is

$$Q(t) = \mathrm{e}^{-6t}\big(-0.059 \cos(8t) - 0.04425 \sin(8t)\big) + \frac{3}{50},$$

and the expression for the current is

$$I(t) = 0.7375\,\mathrm{e}^{-6t} \sin(8t).$$

b)

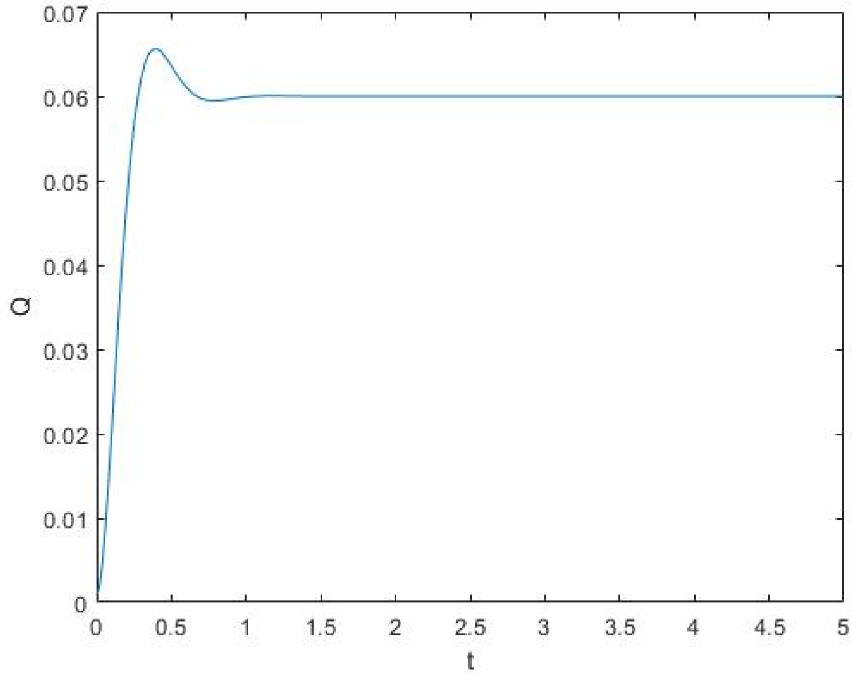


Figure 2: $Q(t) = \mathrm{e}^{-6t}\big(-0.059\cos(8t) - 0.04425\sin(8t)\big) + \frac{3}{50}$

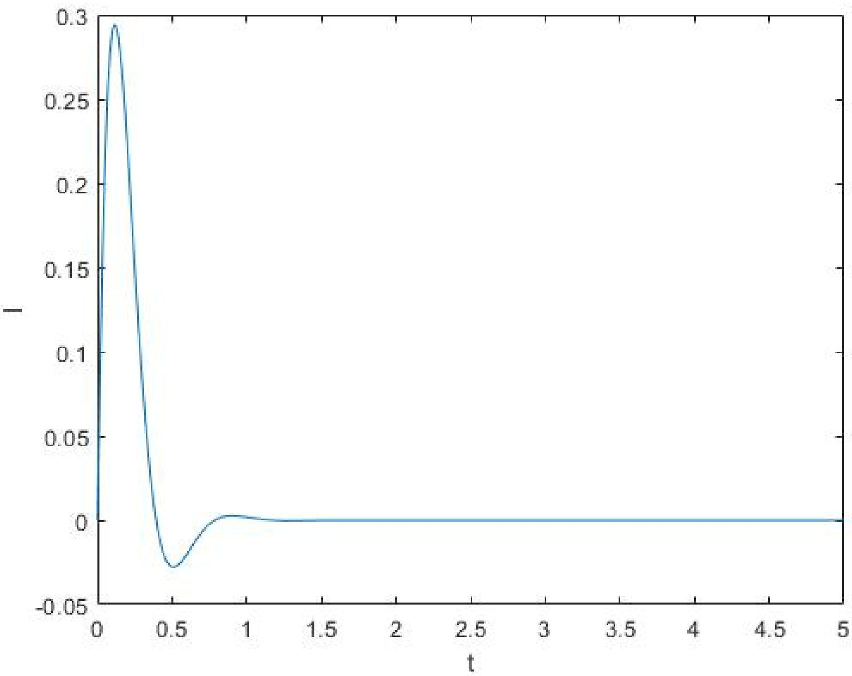


Figure 3: $I(t) = 0.7375\,\mathrm{e}^{-6t}\sin(8t)$

4) Exercise 16

(a) The ODE becomes

$$2\,\frac{d^2Q}{dt^2} + 24\,\frac{dQ}{dt} + 200\,Q = 12\sin(10t). \tag{1}$$

We need to re-compute a particular solution, which can be taken of the form $Q_p(t) = A\cos(10t)+B\sin(10t)$. (This equivalent to the complexification „Ansatz". Let

$$\begin{aligned} Q_p(t) &= A\cos(10t) + B\sin(10t), \\ Q_p'(t) &= -10A\sin(10t) + 10B\cos(10t), \\ Q_p''(t) &= -100A\cos(10t) - 100B\sin(10t). \end{aligned}$$

Substituting this into Equation (1), we have

$$\begin{aligned} 2(-100A\cos(10t) - 100B\sin(10t)) + 24(-10A\sin(10t) + 10B\cos(10t)), \\ +200(A\cos(10t) + B\sin(10t)) = 12\sin(10t), \\ 240B\cos(10t) - 240A\sin(10t) = 12\sin(10t). \end{aligned}$$

So, $A = -0.05$, $B = 0$, which gives $Q_p(t) = -0.05\cos(10t)$.
The general solution is

$$\begin{aligned} Q(t) &= Q_c(t) + Q_p(t) \\ &= e^{-6t}(c_1\cos(8t) + c_2\sin(8t)) - 0.05\cos(10t) \end{aligned}$$

And

$$\begin{aligned} I(t) &= \frac{dQ}{dt} \\ &= e^{-6t}[(-6c_1 + 8c_2)\cos(8t) + (-8c_1 - 6c_2)\sin(8t)] + 0.5\sin(10t) \end{aligned}$$

Imposing the initial conditions $Q(0) = 0.001$ and $I(0) = 0$, we get $c_1 = 0.051$, $c_2 = 0.03825$.
Thus, the formula for the charge is

$$Q(t) = e^{-6t}(0.051\cos(8t) + 0.03825\sin(8t)) - 0.05\cos(10t)$$

(b)

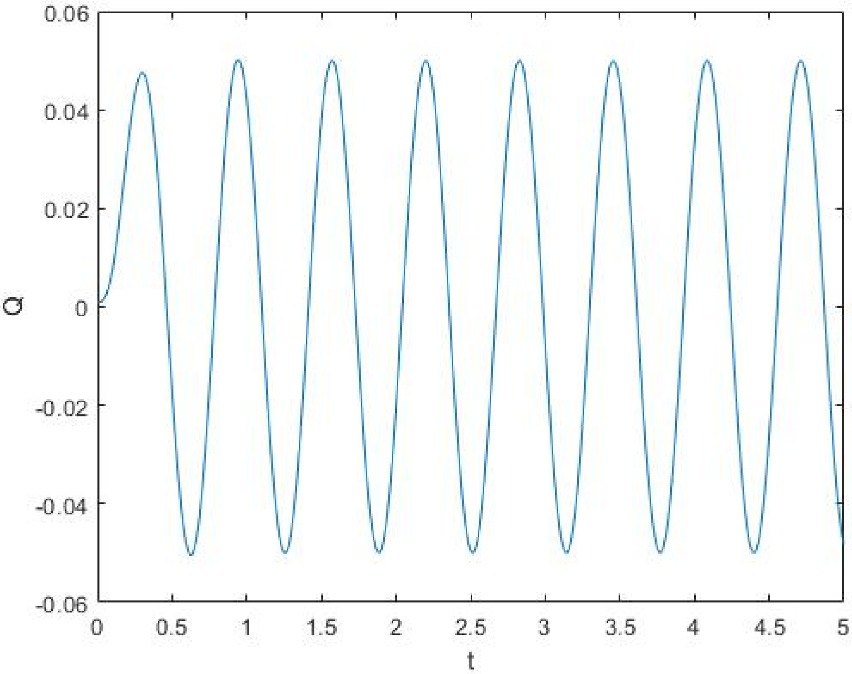


Figure 4: $Q(t) = e^{-6t}(0.051\cos(8t) + 0.03825\sin(8t)) - 0.05\cos(10t)$

42 a) If $\psi(s)$ is a solution of

$$y'' + (\alpha - 1)y' + \beta y = 0,$$

we can use the variable substitution $\psi(s) = \phi(e^s)$ and get the first and second derivative of ψ as

$$\psi'(s) = \phi'(e^s)e^s,$$
$$\psi''(s) = [\phi'(e^s)e^s]' = \phi''(e^s)e^{2s} + \phi'(e^s)e^s.$$

This gives

$$\psi(s)'' + (\alpha - 1)\psi(s)' + \beta\psi(s) = 0,$$
$$[\phi''(e^s)e^{2s} + \phi'(e^s)e^s] + (\alpha - 1)\phi'(e^s)e^s + \beta\phi(s) = 0.$$

This simplifies to

$$e^{2s}\phi''(e^s) + \alpha e^s\phi'(e^s) + \beta\phi(e^s) = 0.$$

Since $t > 0$ is assumed, we can make the variable transformation $t = e^s$, i.e. $s = \ln(t)$, and obtain

$$t^2\phi''(t) + \alpha t\phi'(t) + \beta\phi(t) = 0,$$
$$\phi''(t) + \frac{\alpha}{t}\phi'(t) + \frac{\beta}{t^2}\phi(t) = 0.$$

Therefore, the function $\phi(t)$ is a solution of Equation (1).

Conversely, if $y(t) = \phi(t)$ is a solution of Equation (1), we can make the substitution $t = e^s$ to obtain a function $y(s) = \psi(s)$. The derivatives can be represented as

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}y}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}t} = \frac{1}{t}\frac{\mathrm{d}y}{\mathrm{d}s},$$
$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} = \frac{\mathrm{d}}{\mathrm{d}t}(\frac{1}{t}\frac{\mathrm{d}y}{\mathrm{d}s}) = \frac{1}{t^2}(\frac{\mathrm{d}^2y}{\mathrm{d}s^2} - \frac{\mathrm{d}y}{\mathrm{d}s}).$$

Hence the Euler equation is converted to

$$\frac{\mathrm{d}y^2}{\mathrm{d}s^2} + (\alpha - 1)\frac{\mathrm{d}y}{\mathrm{d}s} + \beta y = 0$$

.

b) 1) When $(\alpha, \beta) = (6, 4)$, the Euler equation can be converted to

$$\frac{\mathrm{d}y^2}{\mathrm{d}s^2} + 5\,\frac{\mathrm{d}y}{\mathrm{d}s} + 4y = 0.$$

The corresponding characteristic equation is $X^2 + 5X + 4 = 0$, and hence the general solution of Equation (2) in this case is

$$y(s) = C_1e^{-s} + C_2e^{-4s}, \quad s \in \mathbb{R}.$$

Therefore, the solution of Equation (1) is

$$y(t) = \frac{C_1}{t} + \frac{C_2}{t^4}, \quad t > 0.$$

2) When $(\alpha, \beta) = (3, 1)$, the Euler equation can be converted to

$$\frac{\mathrm{d}y^2}{\mathrm{d}s^2} + 2\,\frac{\mathrm{d}y}{\mathrm{d}s} + y = 0.$$

The corresponding characteristic equation is $X^2 + 2X + 1 = (X + 1)^2 = 0$, and hence the general solution of Equation (2) in this case is

$$y(s) = C_1e^{-s} + C_2se^{-s}, \quad s \in \mathbb{R}.$$

Therefore, the solution of Equation (1) is

$$y(t) = \frac{C_1}{t} + \frac{C_2 \ln t}{t}, \quad t > 0.$$

43 It is clear that there is a nonzero polynomial $m(X)$ of smallest degree satisfying $m(\mathrm{D})y = 0$, and division of $m(X)$ by the leading coefficient shows that $m(X)$ can be taken as a monic polynomial.
Now suppose $b(\mathrm{D})y = 0$. Long division of $b(X)$ by $m(X)$ gives polynomials $q(X), r(X) \in \mathbb{C}[X]$ with $\deg r(X) < \deg m(X)$ (possibly $r(X) = 0$) and $b(X) = q(X)m(X) + r(X)$. Substituting D shows $r(\mathrm{D})y = b(\mathrm{D})y - q(\mathrm{D})m(\mathrm{D})y = 0$. By minimality of $m(X)$ this is possible only if $r(X) = 0$; $\Longrightarrow m(X)$ divides $b(X)$ in $\mathbb{C}[X]$.
This proves b). For the proof of a) suppose that $m_1(X) \in \mathbb{C}[X]$ is another monic polynomial of smallest degree satisfying $m_1(\mathrm{D})y = 0$. Then b) gives $m(X) \mid m_1(X)$. Since $m(X)$ and $m_1(X)$ have the same degree, they must be constant multiples of each other. The constant must be 1 because $m(X)$, $m_1(X)$ are both monic. Thus $m_1(X) = m(X)$, and $m(X)$ is unique.

44 a) $(\mathrm{D}^2+1)\sin t = (\sin t)'' + \sin t = 0$, $(\mathrm{D}^2+9)\cos(3t) = \cos(3t)'' + 9\cos(3t) = -9\cos(3t) + 9\cos(3t) = 0$,
$\implies (\mathrm{D}^2+1)(\mathrm{D}^2+9)[2\sin t - 3\cos(3t)] = 0$, since $(\mathrm{D}^2+1)(\mathrm{D}^2+9)$ annihilates both $\sin t$ and $\cos(3t)$ and hence any linear combination of these functions. Using the same argument as in the solution of H40b), if follows that $(X^2+1)(X^2+9) = X^4+10X^2+9$ is the monic polynomial of smallest degree annihilating ϕ_1, and hence the "monic" ODE of smallest order having ϕ_1 as solution is $y^{(4)} + 10y'' + 9y = 0$.

b) $\phi_2(t) = \frac{1}{4\mathrm{i}}\left(\mathrm{e}^{\mathrm{i}t} - \mathrm{e}^{-\mathrm{i}t}\right)\left(\mathrm{e}^{3\mathrm{i}t} + \mathrm{e}^{-3\mathrm{i}t}\right) = \frac{1}{4\mathrm{i}}\left(\mathrm{e}^{4\mathrm{i}t} - \mathrm{e}^{-4\mathrm{i}t} - \mathrm{e}^{2\mathrm{i}t} + \mathrm{e}^{-2\mathrm{i}t}\right)$
The corresponding minimum polynomial is $(X-4\mathrm{i})(X+4\mathrm{i})(X-2\mathrm{i})(X+2\mathrm{i}) = (X^2+16)(X^2+4) = X^4 + 20X^2 + 64$, and the desired ODE is $y^{(4)} + 20y'' + 64y = 0$.

c) $\phi_3(t) = -\mathrm{e}^{0\,t} + \frac{1}{2}\,t\,\mathrm{e}^{(-2+\mathrm{i})t} + \frac{1}{2}\,t\,\mathrm{e}^{(-2-\mathrm{i})t}$
The corresponding minimum polynomial is $X(X+2-\mathrm{i})^2(X+2+\mathrm{i})^2 = X(X^2+4X+5)^2 = X^5 + 8\,X^4 + 26\,X^3 + 40\,X^2 + 25\,X$, and the desired ODE is $y^{(5)} + 8y^{(4)} + 26y^{(3)} + 40y'' + 25y' = 0$.

d) The desired minimal ODE is $(\mathrm{D}-1)\mathrm{D}^{2020}y = (\mathrm{D}^{2021} - \mathrm{D}^{2020})y = y^{(2021)} - y^{(2020)} = 0$.

45 a) As observed repeatedly, $a(\mathrm{D})y = 0$ implies $a(\mathrm{D})\mathrm{D}y = a(\mathrm{D})y' = 0$. Thus V is D-invariant.

Let $f = \mathrm{D}|_V$. Then $a(f) = 0$ because of the ODE, and hence the minimum polynomial $\mathrm{m}_f(X)$ of f divides $a(X)$. If there exists a solution $y \in V$ such that $y, y', \dots, y^{(n-1)}$ are linearly independent (i.e., the D-cyclic subspace of V generated by y is equal to V), we can conclude as in b) that $\chi_f(X) = a(X)$. The solution in b) also indicates how to find such a function: Using the EUT, prescribe initial conditions $y(0) = y'(0) = \dots = y^{(n-2)}(0) = 0$, $y^{(n-1)}(0) = 1$ or, in vectorial form, $\big(y(0), y'(0), \dots, y^{(n-1)}(0)\big) = (0, \dots, 0, 1)$. Then $y^{(k)}$ has initial values $\big(y^{k)}(0), y^{(k+1)}(0), \dots, y^{(k+n-1)}(0)\big) = (\underbrace{0, \dots, 0}_{n-1-k}, 1, *, *, \dots)$.
Since the vector of initial values determines the solution (by the EUT), one sees as in b) that $y, y', \dots, y^{(n-1)}$ are linearly independent.

b) $a(\mathrm{S})\mathbf{y} = \mathbf{0}$ implies $a(\mathrm{S})\mathrm{S}\mathbf{y} = \mathrm{S}a(\mathrm{S})\mathbf{y} = \mathrm{S}\mathbf{0} = \mathbf{0}$. Thus V is S-invariant.

Let $f = \mathrm{S}|_V$. Then $a(f) = 0$ because of the recurrence relation, and hence the minimum polynomial $\mathrm{m}_f(X)$ of f divides $a(X)$. If there exists a sequence $\mathbf{y} \in V$ such that $\mathbf{y}$, $\mathrm{S}\mathbf{y}$, …, $\mathrm{S}^{(n-1)}\mathbf{y}$ are linearly independent (i.e., the S-cyclic subspace of V generated by $\mathbf{y}$ is equal to V), we can conclude that $\mathrm{m}_f(X) = a(X)$, and then in turn $\chi_f(X) = a(X)$, because $\deg a(X) = n = \dim V$. A sequence $\mathbf{y} \in V$ with this property is easily found: Just use the last member $\mathrm{e}_{n-1} = (\underbrace{0, \dots, 0}_{n-1}, 1, *, *, \dots)$ of the canonical basis of V. (The remaining entries of e_{n-1} are computed from the first n entries and the recurrence relation.) The matrix formed by the first n coordinates of e_{n-1}, $\mathrm{S}\mathrm{e}_{n-1}$, …, $\mathrm{S}^{n-1}\mathrm{e}_{n-1}$ is triangular with 1's on the main diagonal, showing that these sequences are linearly independent.

46 The discrete analogue of the exponential function, which satisfies $\mathrm{D}\mathrm{e}^t = \mathrm{e}^t$, is the sequence $\mathbf{e} = (1, 1, 1, \dots)$, since $\mathrm{S}\mathbf{x} = \mathbf{x}$ iff $\mathbf{x}$ is a constant sequence. Thus the solution

$y_i = -1$ in the discrete case is rather the analogue of $-\mathrm{e}^t$ than of the constant function $y(t) \equiv -1$ on $\mathbb{R}$, and the observed phenomenon is merely a coincidence.
In fact we should think of $y_{i+2} - y_{i+1} - y_i = 1$ as the discrete analogue of $y'' - y' - y = \mathrm{e}^t$, which has the solution $y(t) = -\mathrm{e}^{-t}$. This is so, because the right-hand side of $y_{i+2} - y_{i+1} - y_i = 1$ is actually a sequence, viz. $(1, 1, 1, \dots)$. In order to complete the picture, consider the recurrence relation

$$y_{i+2} - y_{i+1} - y_i = \delta_{i0} = \begin{cases} 1 & \text{if } i = 0, \\ 0 & \text{if } i \geq 1. \end{cases}$$

This is the true discrete analogue of $y'' - y' - y = 1$. It has the solution $y_0 = -1$, $y_i = 0$ for $i \geq 1$, i.e., $\mathbf{y} = (-1, 0, 0, \dots) = -(\delta_{i0})$, which is the true discrete analogue of $y(t) \equiv -1$. The characteristic polynomial $a(X) = X^2 - X - 1$ of the ODE/recurrence relation satisfies $a(0) = a(1) = -1$. This explains why in both cases the coefficient -1 appears: $\mathrm{D}1 = \mathrm{D}\mathrm{e}^{0\,t} = a(0)1 = -1$, $\mathrm{D}\mathrm{e}^t = a(1)\mathrm{e}^t = -\mathrm{e}^t$. Thus $a(\mathrm{D})y = 1$ is solved by $y(t) \equiv -1$ and $a(\mathrm{D})y = \mathrm{e}^t$ by $y(t) = -\mathrm{e}^t$. The corresponding discrete analogues are: $a(\mathrm{S})\mathbf{y} = (\delta_{i0})$ is solved by $\mathbf{y} = -(\delta_{i0})$ and $a(\mathrm{S})\mathbf{y} = \mathbf{e}$ by $\mathbf{y} = -\mathbf{e}$. The full story is told in `lecture23-27` (in the section "The View from the Top"): The exponential generating function map egf identifies solutions of the linear recurrence relation $a(\mathrm{S})\mathbf{y} = \mathbf{b}$ with solutions of the linear ODE $a(\mathrm{D})y = \mathrm{egf}(\mathbf{b})$.

Differential Equations (Math 285)

H47 Determine a fundamental system of solutions for Bessel's ODE with $p = \frac{1}{2}$,

$$y'' + \frac{1}{t}\,y' + \left(1 - \frac{1}{4t^2}\right) y = 0,$$

using the „Ansatz“ $z = \sqrt{t}\,y$.

H48 The solution to this exercise provides an easy method for computing $e^{\mathbf{A}t}$ for a 2×2 matrix $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $\mathbb{R}^{2\times 2}$ (or $\mathbb{C}^{2\times 2}$). We assume throughout the exercise that $\mathbf{A}$ is not a scalar multiple of the identity matrix $\mathbf{I}_2$.

a) Show $\mathbf{A}^2 = (bc - ad)\mathbf{I}_2 + (a+d)\mathbf{A}$.

b) Use a) to show that there exist uniquely determined functions $c_0, c_1 \colon \mathbb{R} \to \mathbb{R}$ such that

$$e^{\mathbf{A}t} = c_0(t)\mathbf{I}_2 + c_1(t)\mathbf{A} \quad \text{for } t \in \mathbb{R}. \qquad (\star)$$

Further, show that c_0, c_1 are at least twice differentiable.

c) Show that c_0, c_1 solve the homogeneous linear ODE of order 2 with characteristic polynomial $X^2 - (a+d)X + ad - bc$ and satisfy the initial conditions $c_0(0) = 1$, $c_0'(0) = 0$ and $c_1(0) = 0$, $c_1'(0) = 1$.
Hint: Differentiate $(\star)$ twice.

d) By solving the IVP's in c) determine $e^{\mathbf{A}t}$ for $\mathbf{A} = \begin{pmatrix} 0 & 6 \\ 1 & 1 \end{pmatrix}$.

H49 Solve the initial value problem

$$\mathbf{y}' = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix} \mathbf{y} + \begin{pmatrix} t \\ \sin t \end{pmatrix}, \quad \mathbf{y}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

H50 *Optional Exercise*

The function e^t has no zero and satisfies $y' = y$. The function $\sin t$ has no zero in common with its derivative $\cos t$ and satisfies $y'' = -y$. Generalizing this observation, show that a nonzero C^n-function $f\colon I \to \mathbb{R}$ on an interval $I \subseteq \mathbb{R}$ of positive length satisfies an explicit (possibly time-dependent) homogeneous linear ODE of order n if and only if $y, y', \dots, y^{(n-1)}$ have no common zero.
Hint: For the if-part work with the function $t \mapsto f(t)^2 + f'(t)^2 + \cdots + f^{(n-1)}(t)^2$.

Due on Thu April 25, 10 am

The optional exercise is instructive and quite short. It should be handed in on April 25 as well.

47 Using the Ansatz $z = \sqrt{t}\,y$, we have

$$\begin{aligned} \frac{dz}{dt} &= \sqrt{t}\,y' + \frac{1}{2\sqrt{t}}\,y \\ \frac{d^2z}{dt^2} &= \sqrt{t}\,y'' + \frac{1}{\sqrt{t}}\,y' + \left(-\frac{1}{4}\right) t^{-\frac{3}{2}}y \\ \implies \quad 4t^{\frac{3}{2}}\frac{d^2z}{dt^2} &= 4t^2y'' + 4ty' - y \end{aligned}$$

Rewrite the ODE and substituting the above expression, we obtain

$$\begin{aligned} & 4t^2y'' + 4ty' + (4t^2-1)y = 0 \\ \iff \quad & 4t^{\frac{3}{2}}z'' + 4t^2y = 0 \\ \iff \quad & 4t^{\frac{3}{2}}(z + z'') = 0 \\ \iff \quad & z'' + z = 0 \end{aligned}$$

The characteristic equation of $z'' + z = 0$ is $r^2 + 1 = 0$, so that $r_1 = \mathrm{i}$, $r_2 = -\mathrm{i}$.

$$\begin{aligned} \therefore z(t) &= c_1\cos t + c_2 \sin t, \\ \therefore y(t) &= \frac{c_1\cos t}{\sqrt{t}} + \frac{c_2\sin t}{\sqrt{t}}. \end{aligned}$$

Thus a fundamental system of solutions for the ODE is $\frac{\cos t}{\sqrt{t}}$, $\frac{\sin t}{\sqrt{t}}$.

48 a) This follows from the Cayley-Hamilton Theorem, which says $\chi_{\mathbf{A}}(X) = \mathbf{A}^2 - (a+d)\mathbf{A} + (ad-bc)\mathbf{I}_2 = \mathbf{0}$. Here is a direct proof:

$$\begin{aligned} \mathbf{A}^2 &= \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^2+bc & (a+d)b \\ (a+d)c & d^2+bc \end{pmatrix} \\ &= (a+d)\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} a^2+bc-(a+d)a & 0 \\ 0 & d^2+bc-(a+d)d \end{pmatrix} \\ &= (a+d)\mathbf{A} + (bc-ad)\mathbf{I}_2. \end{aligned}$$

b) The relation $\mathbf{A}^2 = \alpha\mathbf{I}_2 + \beta\mathbf{A}$ ($\alpha = bc - ad$, $\beta = a + d$) can be used to express any power $\mathbf{A}^k$ as a linear combination of $\mathbf{I}_2$ and $\mathbf{A}$. It follows that

$$\sum_{k=0}^{n} \frac{t^k}{k!}\mathbf{A}^k = f_n(t)\mathbf{I}_2 + g_n(t)\mathbf{A}$$

for certain functions $f_n, g_n \colon \mathbb{R} \to \mathbb{R}$. Since the left-hand side converges to $\mathrm{e}^{\mathbf{A}t}$, so does the right-hand side, and hence the function sequences (f_n), (g_n) converge (point-wise) to functions f resp. g such that $\mathrm{e}^{\mathbf{A}t} = f(t)\mathbf{I}_2 + g(t)\mathbf{A}$ for $t \in \mathbb{R}$. This representation is unique, since $\mathbf{I}_2$ and $\mathbf{A}$ are assumed to be linearly independent.

From the lecture we know that $\frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{e}^{\mathbf{A}t} = \mathbf{A}\mathrm{e}^{\mathbf{A}t}$. This can be iterated to yield $\frac{\mathrm{d}^2}{\mathrm{d}t^2}\,\mathrm{e}^{\mathbf{A}t} = \mathbf{A}^2\mathrm{e}^{\mathbf{A}t}$ (and likewise $\frac{\mathrm{d}^k}{\mathrm{d}t^k}\,\mathrm{e}^{\mathbf{A}t} = \mathbf{A}^k\mathrm{e}^{\mathbf{A}t}$ for all $k \in \mathbb{N}$). Hence the functions f, g, which are the coordinate functions of $t \mapsto \mathrm{e}^{\mathbf{A}t}$ with respect to the "matrix basis" $\mathbf{I}, \mathbf{A}$, are twice differentiable as well (even of class C^∞).

Thus the assertion holds with $c_0 = f$, $c_1 = g$.

c) Using the observation made about the derivatives of $t \mapsto \mathrm{e}^{\mathbf{A}t}$ in b), we obtain

$$\begin{aligned}\mathrm{e}^{\mathbf{A}t} &= c_0(t)\mathbf{I}_2 + c_1(t)\mathbf{A},\\ \mathbf{A}\mathrm{e}^{\mathbf{A}t} &= c_0'(t)\mathbf{I}_2 + c_1'(t)\mathbf{A},\\ \mathbf{A}^2\mathrm{e}^{\mathbf{A}t} &= c_0''(t)\mathbf{I}_2 + c_1''(t)\mathbf{A}\end{aligned} \qquad (\star\star)$$

for $t \in \mathbb{R}$. Together with a) this yields

$$\begin{aligned}\mathbf{0} &= \mathbf{A}^2\mathrm{e}^{\mathbf{A}t} - (a+d)\mathbf{A}\mathrm{e}^{\mathbf{A}t} + (ad-bc)\mathrm{e}^{\mathbf{A}t}\\ &= \big(c_0''(t) - (a+d)c_0'(t) + (ad-bc)c_0(t)\big)\mathbf{I}_2 + \big(c_1''(t) - (a+d)c_1'(t) + (ad-bc)c_1(t)\big)\mathbf{A}\end{aligned}$$

for all $t \in \mathbb{R}$, which can only hold if the coefficient functions of $\mathbf{I}_2$ and $\mathbf{A}$ vanish, i.e., c_0, c_1 solve the homogeneous linear ODE with characteristic polynomial $X^2 - (a+d)X + ad - bc$. The asserted initial conditions follow by substituting $t = 0$ into the 1st and 2nd equation of $(\star\star)$ and comparing coefficients of $\mathbf{I}_2, \mathbf{A}$.

d) According to a) the given matrix satisfies $\mathbf{A}^2 - \mathbf{A} - 6\mathbf{I}_2 = \mathbf{0}$.
$\Longrightarrow c_0, c_1$ solve $y'' - y' - 6y = 0$, which has characteristic polynomial $X^2 - X - 6 = (X+2)(X-3)$. The general solution of this ODE is $y(t) = a_1\,\mathrm{e}^{-2t} + a_2\,\mathrm{e}^{3t}$, with initial conditions $y(0) = a_1 + a_2$, $y'(0) = -2\,a_1 + 3\,a_2$. A short computation yields $c_0(t) = \frac{3}{5}\,\mathrm{e}^{-2t} + \frac{2}{5}\,\mathrm{e}^{3t}$, $c_1(t) = -\frac{1}{5}\,\mathrm{e}^{-2t} + \frac{1}{5}\,\mathrm{e}^{3t}$, and hence

$$\begin{aligned}\mathrm{e}^{\mathbf{A}t} &= \left(\frac{3}{5}\,\mathrm{e}^{-2t} + \frac{2}{5}\,\mathrm{e}^{3t}\right)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \left(-\frac{1}{5}\,\mathrm{e}^{-2t} + \frac{1}{5}\,\mathrm{e}^{3t}\right)\begin{pmatrix}0 & 6\\ 1 & 1\end{pmatrix}\\ &= \frac{1}{5}\begin{pmatrix}3\,\mathrm{e}^{-2t} + 2\,\mathrm{e}^{3t} & -6\,e^{-2t} + 6\,\mathrm{e}^{3t}\\ -\mathrm{e}^{-2t} + \mathrm{e}^{3t} & 2\,\mathrm{e}^{-2t} + 3\,\mathrm{e}^{3t}\end{pmatrix}.\end{aligned}$$

49 For $\mathbf{A} = \left(\begin{smallmatrix}1 & 2\\ 3 & 6\end{smallmatrix}\right)$ the identity $\mathbf{A}^2 = 7\mathbf{A}$ is easily verified. It is a special case of the Cayley-Hamilton Theorem or of H48 a). From it we obtain $\mathbf{A}^3 = 7\mathbf{A}^2 = 7^2\mathbf{A}$, $\mathbf{A}^4 = 7^2\mathbf{A}^2 = 7^3\mathbf{A}$, etc., and in general $\mathbf{A}^k = 7^{k-1}\mathbf{A}$ for $k \in \mathbb{N}$ by induction.

$$\begin{aligned}\mathrm{e}^{\mathbf{A}t} &= \mathbf{I}_2 + t\,\mathbf{A} + \frac{t^2}{2!}\,\mathbf{A}^2 + \frac{t^3}{3!}\,\mathbf{A}^3 + \cdots\\ &= \mathbf{I}_2 + t\,\mathbf{A} + \frac{7t^2}{2!}\,\mathbf{A} + \frac{7^2t^3}{3!}\,\mathbf{A} + \cdots\\ &= \mathbf{I}_2 + \frac{\mathrm{e}^{7t}-1}{7}\,\mathbf{A} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \begin{pmatrix}\frac{1}{7}\,(\mathrm{e}^{7t}-1) & \frac{2}{7}\,(\mathrm{e}^{7t}-1)\\ \frac{3}{7}\,(\mathrm{e}^{7t}-1) & \frac{6}{7}\,(\mathrm{e}^{7t}-1)\end{pmatrix} = \begin{pmatrix}\frac{1}{7}\,(\mathrm{e}^{7t}+6) & \frac{2}{7}\,(\mathrm{e}^{7t}-1)\\ \frac{3}{7}\,(\mathrm{e}^{7t}-1) & \frac{1}{7}\,(6\,\mathrm{e}^{7t}+1)\end{pmatrix}.\end{aligned}$$

Alternatively, we can proceed as in H48 d) to determine $\mathrm{e}^{\mathbf{A}t}$.
The particular solution $\mathbf{y}(t)$ of the inhomogeneous system satisfying $\mathbf{y}(0) = \mathbf{0}$ is of the

form $\mathbf{y}(t) = \mathrm{e}^{\mathbf{A}t}\mathbf{c}(t)$ with

$$\begin{aligned}
\mathbf{c}(t) &= \int_0^t \mathrm{e}^{-\mathbf{A}s}\,\mathbf{b}(s)\,\mathrm{d}s & \text{(since } \mathbf{c}(0) = \mathbf{0})\\
&= \int_0^t \begin{pmatrix} \frac{1}{7}(\mathrm{e}^{-7s}+6) & \frac{2}{7}(\mathrm{e}^{-7s}-1)\\ \frac{3}{7}(\mathrm{e}^{-7s}-1) & \frac{1}{7}(6\,\mathrm{e}^{-7s}+1)\end{pmatrix}\begin{pmatrix} s\\ \sin s\end{pmatrix}\mathrm{d}s\\
&= \frac{1}{7}\int_0^t \left[\begin{pmatrix} 6 & -2\\ -3 & 1\end{pmatrix} + \mathrm{e}^{-7s}\begin{pmatrix} 1 & 2\\ 3 & 6\end{pmatrix}\right]\begin{pmatrix} s\\ \sin s\end{pmatrix}\mathrm{d}s\\
&= \frac{1}{7}\int_0^t \begin{pmatrix} 6s - 2\sin s + s\,\mathrm{e}^{-7s} + 2\sin s\,\mathrm{e}^{-7s}\\ -3s + \sin s + 3s\,\mathrm{e}^{-7s} + 6\sin s\,\mathrm{e}^{-7s}\end{pmatrix}\mathrm{d}s\\
&= \frac{1}{7}\begin{pmatrix} 3\,t^2 - \frac{1}{49}(7\,t+1)e^{(-7\,t)} - \frac{1}{25}(\cos(t) + 7\,\sin(t))e^{(-7\,t)} + 2\,\cos(t) - \frac{2376}{1225}\\ -\frac{3}{2}\,t^2 - \frac{3}{49}(7\,t+1)e^{(-7\,t)} - \frac{3}{25}(\cos(t) + 7\,\sin(t))e^{(-7\,t)} - \cos(t) - \frac{2376}{1225}\end{pmatrix}.\\
\implies \mathbf{y}(t) = \mathrm{e}^{\mathbf{A}t}\mathbf{c}(t) &= \begin{pmatrix} \frac{3}{7}\,t^2 - \frac{1}{49}\,t + \frac{7}{25}\,\cos(t) + \frac{74}{8575}\,e^{(7\,t)} - \frac{1}{25}\,\sin(t) - \frac{99}{343}\\ -\frac{3}{14}\,t^2 - \frac{3}{49}\,t - \frac{4}{25}\,\cos(t) + \frac{222}{8575}\,e^{(7\,t)} - \frac{3}{25}\,\sin(t) + \frac{46}{343}\end{pmatrix}
\end{aligned}$$

For the last two steps the computer algebra system SageMath was used.
Remark: There is a third way to determine a fundamental matrix of $\mathbf{y}' = \left(\begin{smallmatrix} 1 & 2\\ 3 & 6\end{smallmatrix}\right)\mathbf{y}$ using the eigenvalues and eigenvectors of $\left(\begin{smallmatrix} 1 & 2\\ 3 & 6\end{smallmatrix}\right)$. This will be discussed later in the lecture. An adhoc solution, essentially equivalent to it, is the following: Since $\left(\begin{smallmatrix} 1 & 2\\ 3 & 6\end{smallmatrix}\right)\left(\begin{smallmatrix} 2\\ -1\end{smallmatrix}\right) = \left(\begin{smallmatrix} 0\\ 0\end{smallmatrix}\right)$, there is the constant solution $y_1(t) \equiv \left(\begin{smallmatrix} 2\\ -1\end{smallmatrix}\right)$. Then one needs to guess that a non-constant solution of the form $y_2(t) = \mathrm{e}^{\lambda t}\left(\begin{smallmatrix} v_1\\ v_2\end{smallmatrix}\right)$ exists. Substituting this into the ODE gives $\lambda\,\mathrm{e}^{\lambda t}\left(\begin{smallmatrix} v_1\\ v_2\end{smallmatrix}\right) = y_2'(t) = \left(\begin{smallmatrix} 1 & 2\\ 3 & 6\end{smallmatrix}\right)y_2(t) = \mathrm{e}^{\lambda t}\left(\begin{smallmatrix} v_1+2v_2\\ 3v_1+6v_2\end{smallmatrix}\right)$. Comparing both sides, we obtain $v_1 + 2v_2 = \lambda\,v_1$, $3v_1 + 6v_2 = \lambda\,v_2$. Thus $\lambda v_2 = 3\lambda v_1$ and, since $\lambda \neq 0$, necessarily $v_2 = 3v_1$. Then, using the 1st equation, $\lambda = 7$ since $v_1 \neq 0$. Hence $\mathbf{y}_2(t) = \mathrm{e}^{7t}\left(\begin{smallmatrix} 1\\ 3\end{smallmatrix}\right)$ is a solution, and $\mathbf{\Phi}(t) = \left(\begin{smallmatrix} 2 & \mathrm{e}^{7t}\\ -1 & 3\,\mathrm{e}^{7t}\end{smallmatrix}\right)$ is a fundamental matrix. From there we can either obtain $\mathrm{e}^{\mathbf{A}t}$ using the formula $\mathrm{e}^{\mathbf{A}t} = \mathbf{\Phi}(t)\mathbf{\Phi}(0)^{-1}$ (cf. lecture) and proceed as above, or perform variation of parameters with the fundamental matrix $\mathbf{\Phi}(t)$ in place of $\mathrm{e}^{\mathbf{A}t}$ to obtain a solution of the inhomogeneous system.

50 $\Longrightarrow$: Suppose, by contradiction, that $f^{(n)}(t) = a_0(t)f(t)+a_1(t)f'(t)+\cdots+a_{n-1}(t)f^{(n-1)}(t)$ for all $t \in I$ and $f(t_0) = f'(t_0) = \cdots = f^{(n-1)}(t_0) = 0$ for some $t_0 \in I$. Then both f and the all-zero function on I solve the IVP $y^{(n)} = a_0(t)y+a_1(t)y'+\cdots+a_{n-1}(t)y^{(n-1)} \wedge y(t_0) = y'(t_0) = \cdots = y^{(n-1)}(t_0) = 0$. The Uniqueness Theorem (for linear ODEs, say) then implies that $y \equiv 0$, which contradicts the assumption.
$\Longleftarrow$: Under the given assumption $g(t) = f(t)^2 + f'(t)^2 + \cdots + f^{(n-1)}(t)^2$ is zero-free on I, i.e., we can write

$$1 = \frac{g(t)}{g(t)} = \frac{f(t)^2}{g(t)} + \frac{f'(t)^2}{g(t)} + \cdots + \frac{f^{(n-1)}(t)^2}{g(t)}.$$

Multiplying this identity by $f^{(n)}(t)$ gives

$$f^{(n)}(t) = \frac{f^{(n)}(t)f(t)^2}{g(t)} + \frac{f^{(n)}(t)f'(t)^2}{g(t)} + \cdots + \frac{f^{(n)}(t)f^{(n-1)}(t)^2}{g(t)},$$

which is an explicit homogeneous linear ODE of order n for f with coefficient functions

$$a_0(t) = \frac{f^{(n)}(t)f(t)}{g(t)},\quad a_1(t) = \frac{f^{(n)}(t)f'(t)}{g(t)},\quad \ldots,\quad a_{n-1}(t) = \frac{f^{(n)}(t)f^{(n-1)}(t)}{g(t)}.$$

Differential Equations (Math 285)

H51 Determine the general solution of the following ODE's (two answers suffice):

a) $(2t+1)y''+(4t-2)y'-8y=(6t^2+t-3)\mathrm{e}^t, \quad t>-1/2;$

b) $t^2(1-t)y''+2t(2-t)y'+2(1+t)y=t^2, \quad 0<t<1;$

c) $(t^2-4t+4)y''+(3t-6)y'+2y=t^2+1, \quad t>2.$

Hints: The associated homogeneous ODE in a) has a solution of the form $y(t)=\mathrm{e}^{\alpha t}$ and that in b) a solution of the form $y(t)=t^\beta$ with constants α,β. In both cases a particular solution of the inhomogeneous ODE can be determined by reducing it to a first-order system and using variation of parameters (though this may not be the most economic solution). The ODE in c) is an inhomogeneous Euler equation in disguise.

H52 Prove Leibniz's rule for the n-th derivative of a product: If $f,g\colon I\to\mathbb{R}$ are n-times differentiable then so is $F=fg$, and

$$\mathrm{D}^nF=\sum_{k=0}^{n}\binom{n}{k}(\mathrm{D}^kf)(\mathrm{D}^{n-k}g).$$

H53 *On Hermite Polynomials*

In the lecture the Hermite polynomials $\mathrm{H}_n(X)\in\mathbb{R}[X]$ are defined by $\mathrm{H}_n(t)=(-1)^n\mathrm{e}^{t^2}\mathrm{D}^n[\mathrm{e}^{-t^2}]$ for $t\in\mathbb{R}$ $(n=0,1,2,\dots)$.

a) Show that $t\mapsto\mathrm{H}_n(t)$ is a polynomial function, justifying the definition.

b) Show that $\deg\mathrm{H}_n(X)=n$ and the leading coefficient of $\mathrm{H}_n(X)$ is 2^n.

c) Show that $\mathrm{H}_n(X)$ satisfies the recurrence relation $\mathrm{H}_{n+1}(X)=2X\,\mathrm{H}_n(X)-2n\,\mathrm{H}_{n-1}(X)$, and compute $\mathrm{H}_n(X)$ for $n\leq 6$.

d) Show that $t\mapsto\mathrm{H}_n(t)$ solves Hermite's differential equation $y''-2ty'+2ny=0$.
Hint: The equation is equivalent to $Ly=0$, where $L=\mathrm{D}^2-2t\mathrm{D}+2n\,\mathrm{id}$. Express $L\big[\mathrm{H}_n(t)\big]$ in terms of $\mathrm{D}^n[\mathrm{e}^{-t^2}]$, $\mathrm{D}^{n+1}[\mathrm{e}^{-t^2}]$, $\mathrm{D}^{n+2}[\mathrm{e}^{-t^2}]$, and rewrite the latter using $\mathrm{D}^{n+2}[\mathrm{e}^{-t^2}]=\mathrm{D}^{n+1}[-2t\,\mathrm{e}^{-t^2}]$.

H54 Compute the Taylor series of $z\mapsto 1/(z^2+1)$ at $a=1$ (and, optionally, at $a=1+\mathrm{i}$).
Hint: Proceed as for $z\mapsto 1/(1-z)$ in the lecture and then use partial fractions.

H55 Using power series, solve each of the following initial-value problems:

a) $t(2-t)y''-6(t-1)y'-4y=0, \quad y(1)=1,\ y'(1)=0;$

b) $y''+(t^2+2t+1)y'-(4+4t)y=0, \quad y(-1)=0,\ y'(-1)=1.$

 PLEASE TURN OVER

H56 a) Find 2 linearly independent solutions of $y'' + t^3 y' + 3t^2 y = 0$.

b) Find the first 5 nonzero terms in the Taylor series expansion about $t = 0$ of the solution $y(t)$ of the initial value problem

$$y'' + t^3 y' + 3t^2 y = \mathrm{e}^t, \quad y(0) = y'(0) = 0.$$

H57 *A Problem from Sunday's Lecture*

Suppose (α_n) and (u_n) are sequences of nonnegative real numbers satisfying

$$\begin{aligned} \alpha_n &\leq \sum_{k=0}^{n-1} \frac{M(k+1)}{n(n-1)} \alpha_k \quad (n \geq 2), \\ u_n &= \sum_{k=0}^{n-1} \frac{M(k+1)}{n(n-1)} u_k \quad (n \geq 2), \\ u_0 &= \alpha_0, \ u_1 = \alpha_1 \end{aligned}$$

for some constant $M > 0$.

a) Show $\alpha_n \leq u_n$ for all n.

b) Show $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = 1$.
Hint: Express u_{n+1} in terms of u_n.

c) Is the sequence (u_n) (and hence (α_n) as well) necessarily bounded from above?

H58 For each of the following ODE's, find two linearly independent real solutions.

a) $4xy'' + 3y' - 3y = 0, \quad x \lessgtr 0$;

b) $x^2 y'' - x(1+x)y' + y = 0, \quad x \lessgtr 0$;

c) $x^2 y'' + xy' + (1+x)y = 0, \quad x > 0$.

Due on Thu May 9, 10 am

Exercise H58, which requires a lot of work, can be handed in until Sat May 11, 10 am.

Solutions (prepared by Li Menglu and TH)

51 a) Let $y_1(t) = e^{\alpha t}$, so that $y_1'(t) = \alpha e^{\alpha t}$, $y''(t) = \alpha^2 e^{\alpha t}$. Substituting these into the associated homogeneous ODE, we get

$$(2t+1)\alpha^2 + (4t-2)\alpha - 8 = 0 \Rightarrow (2\alpha^2 + 4\alpha)t + \alpha^2 - 2\alpha - 8 = 0$$

$$\therefore \alpha = -2 \implies y_1(t) = e^{-2t} \text{ is a solution.}$$

Setting $y_2(t) = u(t)e^{-2t}$ and substituting this into the ODE (cf. "order reduction" in the lecture), we get for $u'(t)$ the 1st-order linear ODE

$$u''(t) + \left[2\frac{-2e^{-2t}}{e^{-2t}} + \frac{4t-2}{2t+1}\right]u'(t) = 0 \iff u''(t) + \left(\frac{4t-2}{2t+1} - 4\right)u'(t) = 0.$$

$$\therefore u'(t) = e^{\int -\frac{4t-2}{2t+1}+4dt} = e^{2t}(2t+1)^2 \implies u(t) = \frac{4t^2+1}{2}e^{2t} \implies y_2(t) = \frac{4t^2+1}{2}$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-2t} & 2t^2+\frac{1}{2} \\ -2e^{-2t} & 4t \end{vmatrix} = (2t+1)^2 e^{-2t} \neq 0$$

$\implies y_1(t)$, $y_2(t)$ form a fundamental system of solutions of the homogeneous ODE.

For the inhomogeneous ODE in explicit form we have $b(t) = \frac{6t^2+t-3}{2t+1}\,e^t$. Using variation of parameters for the order-reduced 2×2 system, we need to extract the first coordinate function of

$$\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \int \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ b \end{pmatrix} \mathrm{d}t = \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \int \frac{1}{W} \begin{pmatrix} -y_2 b \\ y_1 b \end{pmatrix} \mathrm{d}t\,.$$

We have

$$\int \frac{1}{W}\begin{pmatrix} -y_2 b \\ y_1 b \end{pmatrix} \mathrm{d}t = \int \begin{pmatrix} -\frac{(2t^2+\frac{1}{2})(6t^2+t-3)}{(2t+1)^3}\,\mathrm{e}^{3t} \\ \frac{(6t^2+t-3)}{(2t+1)^3}\,\mathrm{e}^{t} \end{pmatrix} \mathrm{d}t$$

$$= \begin{pmatrix} \frac{-12t^3+8t^2+5t-4}{6(2t+1)^2}\,\mathrm{e}^{3t} \\ \frac{3t+2}{(2t+1)^2}\,\mathrm{e}^{t} \end{pmatrix}.$$

Remark: For the integration step we have used a computer algebra program. If $r(t)$ is any rational function (quotient of two polynomials) and $a \in \mathbb{C}$ then $r(t)\mathrm{e}^{at}$ can be integrated in finite terms iff there exists a rational function R such that $r(t) = R'(t) + a\,R(t)$; if this is the case then $\int r(t)\mathrm{e}^{at} = R(t)\mathrm{e}^{at}$. (This result is due to Liouville.) Only few rational functions $r(t)$ have this property. In the two cases under consideration one can find $R(t)$ with some effort by using the „Ansatz" $R = u/v^2$, $v(t) = 2t+1$, which the special form of the integrand suggests. The details are omitted.

$$\begin{aligned}
\implies y_p(t) &= \mathrm{e}^{-2t}\,\frac{-12t^3+8t^2+5t-4}{6(2t+1)^2}\,\mathrm{e}^{3t} + \frac{4t^2+1}{2}\,\frac{3t+2}{(2t+1)^2}\,\mathrm{e}^{t} \\
&= \frac{-12t^3+8t^2+5t-4+36t^3+24t^2+9t+6}{6(2t+1)^2}\,\mathrm{e}^{t} \\
&= \frac{24t^3+32t^2+14t+2}{6(2t+1)^2}\,\mathrm{e}^{t} = \left(t+\tfrac{1}{3}\right)\mathrm{e}^{t}
\end{aligned}$$

is a particular solution of the inhomogeneous ODE, and its general solution is

$$y(t) = c_1 e^{-2t} + c_2(4t^2+1) + \left(t+\tfrac{1}{3}\right)\mathrm{e}^t.$$

Remark: A much quicker (but in a way dirty) solution is the following. Using the differential operator

$$L = (2t+1)\mathrm{D}^2 + (4t-2)\mathrm{D} - 8\,\mathrm{id},$$

the inhomogeneous ODE can be written as $L[y] = (6t^2+t-3)\mathrm{e}^t$. It is clear that L maps the space of exponential polynomials of the special form $p(t)\mathrm{e}^t = (p_0 + p_1 t + \cdots + p_d t^d)\mathrm{e}^t$ into itself. Thus we might hope for a particular solution of this form. When determining the images under L of the first few exponential monomials,

$$\begin{aligned} L[\mathrm{e}^t] &= (2t+1)\mathrm{e}^t + (4t-2)\mathrm{e}^t - 8\mathrm{e}^t = (6t-9)\mathrm{e}^t,\\ L[t\,\mathrm{e}^t] &= (2t+1)(t+2)\mathrm{e}^t + (4t-2)(t+1)\mathrm{e}^t - 8t\,\mathrm{e}^t = (6t^2-t)\mathrm{e}^t,\\ &\vdots \end{aligned}$$

we find that

$$6t^2+t-3 = L[t\,\mathrm{e}^t] + \tfrac{1}{3}L[\mathrm{e}^t] = L\left[t\,\mathrm{e}^t + \tfrac{1}{3}\,\mathrm{e}^t\right] = L\left[\left(t+\tfrac{1}{3}\right)\mathrm{e}^t\right].$$

This gives the same particular solution as above. (The general solution is determined in the same way as above.)

b) Let $y_1(t) = t^\beta$, so that $y_1'(t) = \beta t^{\beta-1}$, $y_1''(t) = \beta(\beta-1)t^{\beta-2}$. Substituting these into the associated homogeneous ODE, we get

$$t^2(1-t)\beta(\beta-1)t^{\beta-2} + 2t(2-t)\beta t^{\beta-1} + 2(1+t)t^\beta = 0 \Rightarrow \left[\beta^2+3\beta+2+\left(-\beta^2-\beta+2\right)t\right]t^\beta = 0$$

$$\therefore \beta = -2 \Rightarrow y_1(t) = t^{-2}.$$

Set $y_2(t) = u(t)t^{-2}$ and substitute this into the ODE, we can get

$$u''(t) + \left[2\frac{-2t^{-3}}{t^{-2}} + \frac{2t(2-t)}{t^2(1-t)}\right]u'(t) = 0 \Rightarrow u''(t) + \frac{2}{1-t}u'(t) = 0$$

$$\therefore u'(t) = e^{\int \frac{2}{t-1}dt} = (t-1)^2 \Rightarrow u(t) = \frac{(t-1)^3}{3} \Rightarrow y_2(t) = \frac{(t-1)^3}{3t^2}$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} t^{-2} & \frac{(t-1)^3}{3t^2} \\ -2t^{-3} & (t-1)^2t^{-2} - \frac{2}{3}(t-1)^3t^{-3} \end{vmatrix} = (t-1)^2t^{-4} \neq 0$$

$\Longrightarrow y_1(t)$, $y_2(t)$ form a fundamental system of solutions of the homogeneous ODE.

For the determination of a particular solution of the inhomogeneous ODE we proceed as before, setting $b(t) = \frac{t^2}{t^2(1-t)} = \frac{1}{1-t}$.

$$\begin{aligned} \int \frac{1}{W}\begin{pmatrix} -y_2 b(t) \\ y_1 b(t) \end{pmatrix}\mathrm{d}t &= \int \frac{t^4}{(t-1)^2}\begin{pmatrix} \frac{(t-1)^2}{3t^2} \\ \frac{1}{t^2(1-t)} \end{pmatrix}\mathrm{d}t = \int \begin{pmatrix} \frac{1}{3}t^2 \\ -\frac{t^2}{(t-1)^3} \end{pmatrix} \\ &= \begin{pmatrix} \left(\frac{1}{9}t^3\right) \\ -\ln(t-1) + \frac{2}{t-1} + \frac{1}{2(t-1)^2} \end{pmatrix} \\ y_p(t) &= t^{-2}\frac{1}{9}t^3 + \frac{(t-1)^3}{3t^2}\left(-\ln(t-1) + \frac{2}{t-1} + \frac{1}{2(t-1)^2}\right) \\ &= \frac{2t^3+12t^2-21t+9-6(t-1)^3\ln(t-1)}{18t^2} \end{aligned}$$

In the numerator of $y_p(t)$ we can subtract $2(t-1)^3+11$ to change it into $18t^2-27t-6(t-1)^3\ln(t-1)$, since this amounts to adding a linear combination of $y_1(t)$ and $y_2(t)$ to $y_p(t)$. This leaves the simpler function $t \mapsto 1-\frac{3}{2t}-\frac{(t-1)^3}{3t^2}\log(t-1)$. The general solution of the inhomogeneous ODE is then

$$y(t) = c_1t^{-2} + c_2\frac{(t-1)^3}{t^2} + 1 - \frac{3}{2t} - \frac{(t-1)^3}{3t^2}\log(t-1).$$

c) Writing the associated homogeneous ODE as $(t-2)^2y'' + 3(t-2)y' + 2y = 0$ and setting $t-2 = x$ we get $x^2y'' + 3xy' + 2y = 0$, which is apparently an Euler equation with $\alpha = 3$, $\beta = 2$. The indicial equation is $r^2 + (\alpha-1)r + \beta = r^2 + 2r + 2 = 0$. It has roots $r_1 = -1 + \mathrm{i}$, $r_2 = -1 - \mathrm{i}$, and hence a complex fundamental system of solutions of the (untransformed) homogeneous ODE on $(2,\infty)$ is

$$\begin{aligned} z_1(t) &= (t-2)^{-1+\mathrm{i}} = \mathrm{e}^{\ln(t-2)(-1+\mathrm{i})},\\ z_2(t) &= (t-2)^{-1-\mathrm{i}} = \mathrm{e}^{\ln(t-2)(-1-\mathrm{i})}. \end{aligned}$$

A real fundamental system of solutions—strictly speaking, this is not required—is

$$\begin{aligned} y_1(t) &= \mathrm{Re} z_1(t) = (t-2)^{-1}\cos\ln(t-2),\\ y_2(t) &= \mathrm{Im} z_1(t) = (t-2)^{-1}\sin\ln(t-2). \end{aligned}$$

Since the associated differential operator $L = (t-2)^2\mathrm{D}^2 + 3(t-2)\mathrm{D} + 2\,\mathrm{id}$ maps the space P_2 of (real, say) quadratic polynomials into itself, it is reasonable to guess that there must be a particular solution of the form $y_p(t) = a(t-2)^2 + b(t-2) + c = ax^2 + bx + c$, $x = t-2$. Substituting $y' = 2ax + b$, $y'' = 2a$ into the inhomogeneous ODE, we obtain

$$\begin{aligned} 10ax^2 + 5bx + 2c = x^2 + 4x + 5 &\implies a = \frac{1}{10},\ b = \frac{4}{5},\ c = \frac{5}{2}\\ &\implies y_p(t) = \frac{1}{10}x^2 + \frac{4}{5}x + \frac{5}{2} = \frac{1}{10}t^2 + \frac{2}{5}t + \frac{13}{10}. \end{aligned}$$

The general real (or complex) solution of $(t-2)^2y'' + 3(t-2)y' + 2y = t^2 + 1$ on $(2,\infty)$ is therefore

$$y(t) = c_1(t-2)^{-1}\cos\ln(t-2) + c_2(t-2)^{-1}\sin\ln(t-2) + \frac{1}{10}t^2 + \frac{2}{5}t + \frac{13}{10}$$

with $c_1, c_2 \in \mathbb{R}$ (resp., $c_1, c_2 \in \mathbb{C}$). For the complex solution we could have used the complex fundamental system $z_1(t)$, $z_2(t)$ instead.

52 The formula can easily be proved by induction on n. For $n = 0$ it is trivial, and for $n = 1$ it is $\mathrm{D}(fg) = (\mathrm{D}f)g + f(\mathrm{D}g)$, which is just the product rule of differentiation.

Assuming that the formula holds for n, we obtain

$$\begin{aligned}
\mathrm{D}^{n+1}F &= \mathrm{D}(\mathrm{D}^nF)\\
&= \mathrm{D}\left(\sum_{k=0}^{n}\binom{n}{k}(\mathrm{D}^kf)(\mathrm{D}^{n-k}g\right) && \text{(inductive hypothesis)}\\
&= \sum_{k=0}^{n}\binom{n}{k}\mathrm{D}\big((\mathrm{D}^kf)(\mathrm{D}^{n-k}g)\big) && \text{(linearity of differentiation)}\\
&= \sum_{k=0}^{n}\binom{n}{k}\big((\mathrm{D}^{k+1}f)(\mathrm{D}^{n-k}g)+(\mathrm{D}^kf)(\mathrm{D}^{n-k+1}g)\big) && \text{product rule}\\
&= \sum_{k=0}^{n}\binom{n}{k}(\mathrm{D}^{k+1}f)(\mathrm{D}^{n-k}g)+\sum_{k=0}^{n}\binom{n}{k}(\mathrm{D}^kf)(\mathrm{D}^{n-k+1}g)\\
&= \sum_{k=1}^{n+1}\binom{n}{k-1}(\mathrm{D}^kf)(\mathrm{D}^{n-(k-1)}g)+\sum_{k=0}^{n}\binom{n}{k}(\mathrm{D}^kf)(\mathrm{D}^{n-k+1}g)\\
&= (\mathrm{D}^{n+1}f)(\mathrm{D}^0g)+(\mathrm{D}^0f)(\mathrm{D}^{n+1}g)+\sum_{k=1}^{n}\left[\binom{n}{k-1}+\binom{n}{k}\right](\mathrm{D}^kf)(\mathrm{D}^{n+1-k}g)\\
&= \sum_{k=0}^{n+1}\binom{n+1}{k}(\mathrm{D}^kf)(\mathrm{D}^{n+1-k}g),
\end{aligned}$$

since $\binom{n}{k-1}+\binom{n}{k}=\binom{n+1}{k}$ for $1\le k\le n$ and $\binom{n+1}{0}=\binom{n+1}{n+1}=1$. Thus the formula holds also for $n+1$, and the proof by induction is complete.

53 a) If f is a polynomial function then

$$\mathrm{D}\left[f(t)\mathrm{e}^{-t^2}\right]=f'(t)\mathrm{e}^{-t^2}+f(t)\mathrm{e}^{-t^2}(-2t)=\big(f'(t)-2t\,f(t)\big)\mathrm{e}^{-t^2}=F(t)\mathrm{e}^{-t^2}, \qquad \text{(H)}$$

where $F(t)=f'(t)-2t\,f(t)$ is also a polynomial function. Starting with $f(t)=f_0(t)=1$, it follows by induction that $\mathrm{D}^n[\mathrm{e}^{-t^2}]=f_n(t)\mathrm{e}^{-t^2}$ for some polynomial function f_n. Hence $\mathrm{H}_n(t)=(-1)^n\mathrm{e}^{t^2}[\mathrm{D}^n\mathrm{e}^{-t^2}]=(-1)^nf_n(t)$ is also a polynomial function.

b) From (H) we see that f_n has degree n and leading coefficient $(-2)^n$. Hence $\mathrm{H}_n(t)$ has degree n as well and leading coefficient 2^n.

c) We have

$$\begin{aligned}
\mathrm{H}_{n+1}(t) &= (-1)^{n+1}\mathrm{e}^{t^2}\mathrm{D}^n\left[\mathrm{D}\mathrm{e}^{-t^2}\right]=(-1)^{n+1}\mathrm{e}^{t^2}\mathrm{D}^n\left[-2t\,\mathrm{e}^{-t^2}\right]=2(-1)^n\mathrm{e}^{t^2}\mathrm{D}^n\left[t\,\mathrm{e}^{-t^2}\right]\\
&= 2(-1)^n\mathrm{e}^{t^2}\big(t\mathrm{D}^n[\mathrm{e}^{-t^2}]+n\mathrm{D}^{n-1}[\mathrm{e}^{-t^2}]\big) && \text{(by Leibniz' formula)}\\
&= 2t(-1)^n\mathrm{e}^{t^2}\mathrm{D}^n[\mathrm{e}^{-t^2}]+2n(-1)^n\mathrm{e}^{t^2}\mathrm{D}^{n-1}[\mathrm{e}^{-t^2}]=2t\,\mathrm{H}_n(t)-2n\,\mathrm{H}_{n-1}(t).
\end{aligned}$$

This proves the recursion formula in view of the 1-1 correspondence between polynomials in $\mathbb{R}[X]$ and polynomial functions on $\mathbb{R}$.

Together with $\mathrm{H}_0(t) = (-1)^0 \mathrm{e}^{t^2}(\mathrm{e}^{-t^2}) = 1$, $\mathrm{H}_1(t) = (-1)^1 \mathrm{e}^{t^2}(-2t\,\mathrm{e}^{-t^2}) = 2t$ the recursion formula gives

$$\begin{aligned}
&\mathrm{H}_0(X) = 1,\\
&\mathrm{H}_1(X) = 2X,\\
&\mathrm{H}_2(X) = 2X(2X) - 2\cdot 1 = 4\,X^2 - 2,\\
&\mathrm{H}_3(X) = 2X(4X^2-2) - 4(2X) = 8\,X^3 - 12\,X,\\
&\mathrm{H}_4(X) = 2X(8X^3-12X) - 6(4X^2-2) = 16\,X^4 - 48\,X^2 + 12,\\
&\mathrm{H}_5(X) = 2X(16\,X^4 - 48\,X^2 + 12) - 8(8\,X^3 - 12\,X) = 32\,X^5 - 160\,X^3 + 120\,X,\\
&\mathrm{H}_6(X) = 2X(32X^5 - 160X^3 + 120X) - 10(16X^4 - 48X^2 + 12) = 64X^6 - 480\,X^4 + 720\,X^2 - 120.
\end{aligned}$$

d) We have

$$\begin{aligned}
L\big[\mathrm{H}_n(t)\big] &= (-1)^n \mathrm{D}^2\big[\mathrm{e}^{t^2}\mathrm{D}^n[\mathrm{e}^{-t^2}]\big] - 2t(-1)^n\mathrm{D}\big[\mathrm{e}^{t^2}\mathrm{D}^n[\mathrm{e}^{-t^2}]\big] + 2n(-1)^n\mathrm{e}^{t^2}\mathrm{D}^n[\mathrm{e}^{-t^2}],\\
(-1)^n L\big[\mathrm{H}_n(t)\big] &= \mathrm{e}^{t^2}\mathrm{D}^{n+2}[\mathrm{e}^{-t^2}] + 2\,2t\mathrm{e}^{t^2}\mathrm{D}^{n+1}[\mathrm{e}^{-t^2}] + (2+4t^2)\mathrm{e}^{t^2}\mathrm{D}^n[\mathrm{e}^{-t^2}]\\
&\quad - 2t\mathrm{e}^{t^2}\mathrm{D}^{n+1}[\mathrm{e}^{-t^2}] - 2t\,2t\mathrm{e}^{t^2}\mathrm{D}^n[\mathrm{e}^{-t^2}] + 2n\,\mathrm{e}^{t^2}\mathrm{D}^n[\mathrm{e}^{-t^2}],\\
(-1)^n\mathrm{e}^{-t^2} L\big[\mathrm{H}_n(t)\big] &= \mathrm{D}^{n+2}[\mathrm{e}^{-t^2}] + 2t\mathrm{D}^{n+1}[\mathrm{e}^{-t^2}] + 2(n+1)\mathrm{D}^n[\mathrm{e}^{-t^2}].
\end{aligned}$$

On the other hand, we also have

$$\mathrm{D}^{n+2}[\mathrm{e}^{-t^2}] = \mathrm{D}^{n+1}[-2t\,\mathrm{e}^{-t^2}] = -2t\mathrm{D}^{n+1}[\mathrm{e}^{-t^2}] - 2(n+1)\mathrm{D}^n[\mathrm{e}^{-t^2}].$$

$\Longrightarrow (-1)^n\mathrm{e}^{-t^2} L\big[\mathrm{H}_n(t)\big] = 0 \Longrightarrow L\big[\mathrm{H}_n(t)\big] = 0$, as desired.

54 $\underline{a = 1:}$

$$\begin{aligned}
\frac{1}{z^2+1} &= \frac{1}{(z-\mathrm{i})(z+\mathrm{i})} = \frac{1}{2\mathrm{i}}\left(\frac{1}{z-\mathrm{i}} - \frac{1}{z+\mathrm{i}}\right)\\
&= \frac{1}{2\mathrm{i}}\left(\frac{1}{z-1+1-\mathrm{i}} - \frac{1}{z-1+1+\mathrm{i}}\right)\\
&= \frac{1}{2\mathrm{i}}\left[\sum_{n=0}^{\infty}(-1)^n\frac{(z-1)^n}{(1-\mathrm{i})^{n+1}} - \sum_{n=0}^{\infty}(-1)^n\frac{(z-1)^n}{(1+\mathrm{i})^{n+1}}\right]\\
&= \sum_{n=0}^{\infty} b_n (z-1)^n
\end{aligned}$$

with

$$\begin{aligned}
b_n &= \frac{(-1)^n}{2^{(n+1)/2}}\,\frac{(\mathrm{e}^{\mathrm{i}\pi/4})^{n+1} - (\mathrm{e}^{-\mathrm{i}\pi/4})^{n+1}}{2\mathrm{i}} = \frac{(-1)^n}{2^{(n+1)/2}}\sin\frac{(n+1)\pi}{4}\\
&= \begin{cases}
2^{-n/2-1} & \text{if } n = 8k,\ 8k+2,\\
-2^{-(n+1)/2} & \text{if } n = 8k+1,\\
0 & \text{if } n = 8k+3,\ 8k+7,\\
-2^{-n/2-1} & \text{if } n = 8k+4,\ 8k+6,\\
2^{-(n+1)/2} & \text{if } n = 8k+5.
\end{cases}
\end{aligned}$$

This can also be written as

$$\frac{1}{z^2+1}=\sum_{k=0}^{\infty}\frac{(z-1)^{8k}}{16^k}\left(\frac{1}{2}-\frac{(z-1)}{2}+\frac{(z-1)^2}{4}-\frac{(z-1)^4}{8}+\frac{(z-1)^5}{8}-\frac{(z-1)^6}{16}\right).$$

and shows the known fact that $\sum_{n=0}^{\infty} b_n(z-1)^n$ has radius of convergence $\sqrt{2}$ (the distance from 1 to the singularities $\pm\mathrm{i}$ of $1/(z^2+1)$).

$\underline{a=1+\mathrm{i}:}$ Since $\frac{1}{(z-a)(z-b)}=\frac{1}{a-b}\left(\frac{1}{z-a}-\frac{1}{z-b}\right)$, we obtain

$$\begin{aligned}\frac{1}{z^2+1}&=\frac{1}{(z-1-\mathrm{i}+1)(z-1-\mathrm{i}+1+2\mathrm{i}}=\frac{\mathrm{i}}{2}\left(\frac{1}{z-1-\mathrm{i}+1}-\frac{1}{z-1-\mathrm{i}+1+2\mathrm{i}}\right)\\&=\frac{\mathrm{i}}{2}\left[\sum_{n=0}^{\infty}(-1)^n(z-1-\mathrm{i})^n-\sum_{n=0}^{\infty}(-1)^n\frac{(z-1-\mathrm{i})^n}{(1+2\mathrm{i})^{n+1}}\right]\\&=\sum_{n=0}^{\infty}c_n(z-1-\mathrm{i})^n\end{aligned}$$

with

$$c_n=\frac{(-1)^n\,\mathrm{i}}{2}\left(1-\frac{(1-2\mathrm{i})^{n+1}}{5^{n+1}}\right)=\frac{(-1)^n\,\mathrm{i}}{2}\left(1-\frac{\left(\frac{1-2\mathrm{i}}{\sqrt{5}}\right)^{n+1}}{5^{(n+1)/2}}\right).$$

Since $\left|\frac{1-2\mathrm{i}}{\sqrt{5}}\right|=1$, the last representation shows $c_n\simeq(-1)^n\,\mathrm{i}/2$ for $n\to\infty$, implying the known fact that $\sum_{n=0}^{\infty}c_n(z-1-\mathrm{i})^n$ has radius of convergence 1 (the distance from $1+\mathrm{i}$ to the nearest singularity i of $1/(z^2+1)$).

55 a) We look for a solution in the form of a power series about $t_0=1$. The series has the form

$$y(t)=\sum_{n=0}^{\infty}a_n(t-1)^n.$$

The point $t_0=1$ is an ordinary point of the differential equation, so the power series solution will be analytic at this point. Moreover, since the coefficient functions $p(t)=\frac{-6(t-1)}{t(2-t)}$, $q(t)=\frac{-4}{t(2-t)}$ of the corresponding explicit ODE have their singularities, viz. $t=0$ and $t=2$, at distance 1 from t_0, the radius of convergence of the power series will be at least 1, and $y(t)$ will solve the ODE on $(-1,1)$.

Differentiating the equation term by term, we obtain that

$$\begin{aligned}y'(t)&=\sum_{n=1}^{\infty}a_n n(t-1)^{n-1},\\y''(t)&=\sum_{n=2}^{\infty}a_n n(n-1)(t-1)^{n-2}.\end{aligned}$$

Substituting the above series into the original equation gives

$$t(2-t)\sum_{n=2}^{\infty}a_n n(n-1)(t-1)^{n-2}-6(t-1)\sum_{n=1}^{\infty}a_n n(t-1)^{n-1}-4\sum_{n=0}^{\infty}a_n(t-1)^n=0.$$

Rewrite the series so that they display the same generic term and using $t(2-t) = 1-(t-1)^2$ gives

$$\sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)(t-1)^n - \sum_{n=2}^{\infty} a_n n(n-1)(t-1)^n - 6\sum_{n=1}^{\infty} a_n n(t-1)^n - \\ - 4\sum_{n=0}^{\infty} a_n(t-1)^n = 0,$$

which can be simplified to

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1)a_{n+2} - (n^2+5n+4)a_n\right](t-1)^n = 0.$$

Hence the coefficients a_n must satisfy the recurrence relation

$$a_{n+2} = \frac{n^2+5n+4)}{(n+2)(n+1)}a_n = \frac{n+4}{n+2}a_n, \quad n = 0,1,2,3,4,\dots$$

According to the initial conditions,

$$a_0 = y(1) = 1, \quad a_1 = y'(1) = 0.$$

The solution is $a_{2k+1} = 0$ for $k = 0,1,2,\dots$ and

$$a_{2k} = \frac{2k+2}{2k}a_{2k-2} = \cdots = \frac{2k+2}{2k}\,\frac{2k}{2k-2}\cdots\frac{4}{2}a_0 = \frac{2k+2}{2} = k+1 \quad \text{for } k = 0,1,2,\dots.$$

Substituting these coefficients into the original series, the solution of the IVP is

$$y(t) = \sum_{k=0}^{\infty}(k+1)(t-1)^{2k}, \quad -1 < t < 1.$$

The radius of convergence of this power series is obviously 1.

Remark: Making the variable transformation $x = t-1$ early on saves some writing (but otherwise leads to the same solution, of course).

b) We look for a solution in the form of a power series about $t_0 = -1$. The series has the form

$$y = \sum_{n=0}^{\infty} a_n(t+1)^n.$$

The point $t_0 = -1$ is an ordinary point of the differential equation, and the coefficient functions $p(t) = (t+1)^2$, $q(t) = -4(t+1)$ are polynomials. Hence the power series will have radius of convergence ∞ and $y(t)$ will be defined and solve the ODE on $\mathbb{R}$. Proceeding as before, we obtain

$$\sum_{n=2}^{\infty} a_n n(n-1)(t+1)^{n-2} + (t+1)^2\sum_{n=1}^{\infty} a_n n(t+1)^{n-1} - 4(t+1)\sum_{n=0}^{\infty} a_n(t+1)^n = 0,$$

$$\sum_{n=2}^{\infty} a_n n(n-1)(t+1)^{n-2} + \sum_{n=1}^{\infty} a_n n(t+1)^{n+1} - 4\sum_{n=0}^{\infty} a_n(t+1)^{n+1} = 0,$$

$$2a_2 + \sum_{n=0}^{\infty}\left[(n+3)(n+2)a_{n+3} + (n-4)a_n\right](t+1)^{n+1} = 0.$$

Hence the coefficients a_n must satisfy

$$a_2 = 0, \quad a_{n+3} = -\frac{n-4}{(n+3)(n+2)}\,a_n \quad \text{for } n = 0, 1, 2, 3, \ldots.$$

The initial conditions are

$$a_0 = y(-1) = 0, \quad a_1 = y'(-1) = 1.$$

Hence $a_0 = a_3 = a_6 = \cdots = 0$, $a_2 = a_5 = a_8 = \cdots = 0$,

$$\begin{aligned} a_4 &= -\frac{1-4}{(1+3)(1+2)}\,a_1 = \frac{3}{12} = \frac{1}{4}, \\ a_7 &= -\frac{4-4}{(4+3)(4+2)}\,a_4 = 0, \end{aligned}$$

and $a_{10} = a_{13} = \cdots = 0$ as well. Substituting these coefficients into the original series, the solution of the IVP is

$$y = (t+1) + \frac{1}{4}(t+1)^4, \quad t \in \mathbb{R}.$$

56 a) As in H55b) solutions at $t_0 = 0$ must be analytic and exist on the whole real line. The power series „Ansatz“ $y(t) = \sum_{n=0}^{\infty} a_n t^n$ yields

$$\sum_{n=2}^{\infty} a_n n(n-1)t^{n-2} + t^3 \sum_{n=1}^{\infty} a_n n t^{n-1} + 3t^2 \sum_{n=0}^{\infty} a_n t^n = 0,$$

$$\sum_{n=-2}^{\infty} (n+4)(n+3)a_{n+4}t^{n+2} + \sum_{n=0}^{\infty} n a_n t^{n+2} + 3\sum_{n=0}^{\infty} a_n t^{n+2} = 0,$$

$$2a_2 + 6a_3 t + \sum_{n=0}^{\infty} \left[(n+4)(n+3)a_{n+4} + (n+3)a_n\right] t^{n+2} = 0.$$

Hence the coefficients a_n satisfy

$$a_2 = a_3 = 0, \quad a_{n+4} = -\frac{1}{n+4}\,a_n \quad \text{for } n = 0, 1, 2, 3, \ldots.$$

Two linearly independent solutions are obtained by setting $(a_0, a_1) = (1, 0)$ and $(0, 1)$, respectively, i.e.,

$$\begin{aligned} y_1(t) &= 1 - \frac{t^4}{4} + \frac{t^8}{4\cdot 8} - \frac{t^{12}}{4\cdot 8\cdot 12} \pm \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k}{4^k k!}\,t^{4k}, \\ y_2(t) &= t - \frac{t^5}{5} + \frac{t^9}{5\cdot 9} - \frac{t^{13}}{5\cdot 9\cdot 13} \pm \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k}{5\cdot 9\cdot 13\cdots(4k+1)}\,t^{4k+1}. \end{aligned}$$

b) The right-hand side of the equation can be expressed using Taylor series as

$$e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!} = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{x^4}{4!} + \cdots.$$

Inserting this series into the ODE and using the initial conditions $a_0 = y(0) = 0$, $a_1 = y'(0) = 0$, changes the homogeneous recurrence relation in a) to the inhomogeneous recurrence relation $a_0 = a_1 = 0$, $a_2 = \frac{1}{2}$, $a_3 = \frac{1}{6}$, and

$$a_{n+4} = -\frac{1}{n+4}\,a_n + \frac{1}{(n+4)!} \quad \text{for } n = 0, 1, 2, 3, \dots$$

The latter is obtained from equating coefficients at t^{n+2}, which gives $(n+4)(n+3)a_{n+4} + (n+3)a_n = \frac{1}{(n+2)!}$. The first few terms in the Taylor series expansion about $t = 0$ of the solution are then

$$y(t) = \frac{1}{2}\,t^2 + \frac{1}{6}\,t^3 + \frac{1}{24}\,t^4 + \frac{1}{120}\,t^5 - \frac{59}{6!}\,t^6 - \frac{119}{7!}\,t^7 - \frac{209}{8!}\,t^8 - \frac{335}{9!}\,t^9 + \frac{29737}{10!}\,t^{10} + \cdots.$$

(We asked for the "first 5 nonzero terms", because $a_6 = -\frac{59}{6!}$ disproves the apparent pattern $a_n = \frac{1}{n!}$, which holds for $n = 2, 3, 4, 5$.)

57 a) The assertion is trivially true for $n = 0, 1$. For $n \geq 2$ we may assume by induction that $\alpha_k \leq u_k$ for $0 \leq k < n$.

$$\implies \alpha_n \leq \sum_{k=0}^{n-1} \frac{M(k+1)}{n(n-1)}\alpha_k \leq \sum_{k=0}^{n-1} \frac{M(k+1)}{n(n-1)}u_k = u_n.$$

b) We have

$$\begin{aligned} u_{n+1} &= \frac{1}{(n+1)n}\sum_{k=0}^{n} M(k+1)u_k = \frac{1}{(n+1)n}\left(\sum_{k=0}^{n-1} M(k+1)u_k + M(n+1)u_n\right) \\ &= \frac{n(n-1)u_n + M(n+1)u_n}{(n+1)n} = \frac{n(n-1) + M(n+1)}{(n+1)n}\,u_n \quad \text{for } n \geq 2. \end{aligned}$$

It follows that

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{n(n-1) + M(n+1)}{(n+1)n} = \lim_{n\to\infty} \frac{n^2 + (M-1)n + M}{n^2 + n} = 1.$$

c) The answer is "No". For $M \leq 2$ the sequence (u_n) remains bounded, but for $M > 2$ it diverges to $+\infty$ (except in the trivial case $u_0 = u_1 = 0$, in which $u_n = 0$ for all n).

The sum of the coefficients in the definition of u_n is $\frac{Mn(n+1)/2}{n(n-1)} = \frac{M(n+1)}{2(n-1)} \approx M/2$ for large n. For $M < 2$ the coefficient sum is ≤ 1 for large n, and one can prove by induction that (u_n) is bounded. (We had a similar example in the lecture.)

We will now show that if u_0, u_1 are not both zero and $M > 2$ then (u_n) is unbounded. Applying the formula for u_{n+1}/u_n repeatedly, we have

$$u_{n+1} = u_2 \prod_{k=2}^{n} \frac{k(k-1) + M(k+1)}{(k+1)k}.$$

This says that the numbers u_n are the partial products of the infinite product

$$\prod_{n=2}^{\infty} \frac{n(n-1) + M(n+1)}{(n+1)n}.$$

It is known that an infinite product $\prod_{n=1}^{\infty}(1+b_n)$ with $b_n \geq 0$ converges (equivalently, is bounded) iff the series $\sum_{n=1}^{\infty} b_n$ converges. (In what follows we need only the implication $\Longrightarrow$, which is clear from $\prod_{k=1}^{n}(1+b_k) \geq 1+\sum_{k=1}^{n} b_k$.) Since

$$\frac{n(n-1)+M(n+1)}{(n+1)n} = 1+\frac{(M-2)n+M}{n^2+n} > 1+\frac{(M-2)n+M-2}{n^2+n} = 1+\frac{M-2}{n},$$

the divergence of the harmonic series implies for $M>2$ that $\lim_{n\to\infty} u_n = \infty$ as well. (For $M=2$ the fact about infinite products quoted above shows that (u_n) converges in $\mathbb{R}$, since this is true of the series $\sum_{n=1}^{\infty}\frac{1}{n^2+n}$.)

58 a) Rewriting the ODE as

$$y'' + \frac{3}{4x}y' - \frac{3}{4x}y = 0,$$

we see that $x=0$ is a regular singular point and

$$p_0 = \lim_{x\to 0} x\frac{3}{4x} = \frac{3}{4}, \quad q_0 = \lim_{x\to 0} x^2\frac{-3}{4x} = 0$$

. $\Longrightarrow$ The indicial equation is

$$r^2 + (p_0-1)r + q_0 = r^2 - \frac{1}{4}r = 0$$

.

$\Longrightarrow$ The exponents at the singularity $x=0$ are $r_1=0$, $r_2=\frac{1}{4}$. Since r_1-r_2 is not an integer, there must be solutions $y_1(x)$, $y_2(x)$ on $(0,\infty)$ of the form

$$y_1(x) = 1+\sum_{n=0}^{\infty} a_n x^n, \quad y_2(x) = x^{\frac{1}{4}}\left(1+\sum_{n=0}^{\infty} a_n x^n\right).$$

In terms of the rational functions $a_n(r)$ defined in the lecture and textbook, the coefficients of $y_1(x)$, $y_2(x)$ are $a_n = a_n(0)$ and $a_n = a_n(1/4)$, respectively. (We use 'a_n' for both, in order to be compatible with the notation used in [BDM17], Theorem 5.6.1.)

i) $r_1 = 0$:

$$\begin{aligned}
y_1 &= \sum_{n=0}^{\infty} a_n x^n \\
y_1' &= \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty}(n+1)a_{n+1}x^n \\
xy_1'' &= x\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=1}^{\infty}(n+1)n a_{n+1}x^n
\end{aligned}$$

Substituting these into the ODE, we get

$$4\sum_{n=1}^{\infty}(n+1)na_{n+1}x^n + 3\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n - 3\sum_{n=0}^{\infty}a_n x^n = 0$$

$$\Longrightarrow \quad 3a_1 - 3a_0 + \sum_{n=1}^{\infty}\left\{\left[4n(n+1)+3(n+1)\right]a_{n+1} - 3a_n\right\}x^n = 0$$

$$\Longrightarrow \quad a_1 = a_0 \quad \text{and} \quad a_{n+1} = \frac{3}{(4n+3)(n+1)}a_n \quad \text{for} \quad n \geq 1.$$

Setting $a_0 = 1$, we have

$$\begin{aligned}
y_1(x) &= 1 + x + \frac{3}{7\cdot 2}\,x^2 + \frac{3^2}{7\cdot 2\cdot 11\cdot 3}\,x^3 + \cdots \\
&= 1 + x + \sum_{n=2}^{\infty} \frac{3^{n-1}}{7\cdot 11\cdots(4n-1)\cdot n!}\,x^n \\
&= 1 + \sum_{n=1}^{\infty} \frac{3^{n}}{3\cdot 7\cdot 11\cdots(4n-1)\cdot n!}\,x^n \\
&= \sum_{n=0}^{\infty} \frac{3^{n}}{3\cdot 7\cdot 11\cdots(4n-1)\cdot n!}\,x^n,
\end{aligned}$$

using the convention that $\prod_{n=1}^{0}(4n-1) = 1$ ("empty product").

ii) $r_2 = \frac{1}{4}$:

$$\begin{aligned}
y_2 &= x^{\frac14}\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+\frac14} \\
y_2' &= \sum_{n=0}^{\infty}\left(n+\frac14\right) a_n x^{n-\frac34} = \sum_{n=-1}^{\infty}\left(n+\frac54\right) a_{n+1} x^{n+\frac14} \\
xy_2'' &= \sum_{n=0}^{\infty}\left(n+\frac14\right)\left(n-\frac34\right) a_n x^{n-\frac34} = \sum_{n=-1}^{\infty}\left(n+\frac54\right)\left(n+\frac14\right) a_{n+1} x^{n+\frac14}
\end{aligned}$$

Substituting these into the ODE, the coefficient of $x^{-3/4}$ vanishes by construction, and we get

$$\sum_{n=0}^{\infty}\left\{\left[4\left(n+\frac54\right)\left(n+\frac14\right) + 3\left(n+\frac54\right)\right] a_{n+1} - 3a_n\right\} x^{n+\frac14} = 0$$

$$\Longrightarrow \quad a_{n+1} = \frac{3\,a_n}{4\left(n+\frac54\right)\left(n+\frac14\right) + 3\left(n+\frac54\right)} = \frac{3\,a_n}{(4n+5)(n+1)} \quad \text{for} \quad n \geq 0.$$

Setting $a_0 = 1$, we obtain

$$y_2(x) = x^{\frac14} + \sum_{n=1}^{\infty}\frac{3^n}{5\cdot 9\cdots(4n+1)\cdot n!}\,x^{n+\frac14} = \sum_{n=0}^{\infty}\frac{3^n}{5\cdot 9\cdots(4n+1)\cdot n!}\,x^{n+\frac14}.$$

As shown in the lecture, $y_1(x)$ and $y_2(x)$ are linearly independent. This is also clear from the fact that $y_1(x)$ is analytic at $x = 0$ and $y_2(x) = x^{1/4} \times$ "nonzero analytic" is not.

As discussed in the lecture (or see Theorem 5.6.1 in [BDM17], p. 227), a fundamental system of solutions on $(-\infty, 0)$ is obtained by replacing the fractional part x^r (if any) in the solutions by $(-x)^r = |x|^r$. This doesn't affect $y_1(x)$ ($y_1(x)$ is analytic on $\mathbb{R}$ and hence solves the ODE on $\mathbb{R}$), but $y_2(x)$ is changed to

$$y_2^-(x) = (-x)^{\frac14}\sum_{n=0}^{\infty}\frac{3^n}{5\cdot 9\cdots(4n+1)\cdot n!}\,x^n, \quad x \in (-\infty, 0).$$

b) Rewriting the ODE as

$$y'' - \left(1 + \frac{1}{x}\right) y' + \frac{1}{x^2}\, y = 0,$$

we see that $x = 0$ is a regular singular point with $p_0 = -1$, $q_0 = 1$.
$\implies$ The indicial equation is

$$r^2 + (p_0 - 1)r + q_0 = (r-1)^2 = 0$$

.
$\implies$ The exponents at the singularity $x = 0$ are $r_1 = r_2 = 1$. Thus there must be solutions $y_1(x)$, $y_2(x)$ on $(0, \infty)$ of the form

$$y_1(x) = 1 + \sum_{n=1}^{\infty} a_n x^{n+1}, \qquad y_2(x) = y_1(x) \ln x + \sum_{n=1}^{\infty} b_n x^{n+1}.$$

i) $r_1 = 1$:

$$\begin{aligned}
y_1 &= \sum_{n=0}^{\infty} a_n x^{n+1}, \\
y_1' &= \sum_{n=0}^{\infty} (n+1) a_n x^n, \\
x(1+x)y_1' &= \sum_{n=0}^{\infty} (n+1) a_n x^{n+1} + \sum_{n=0}^{\infty} (n+1) a_n x^{n+2} \\
&= \sum_{n=0}^{\infty} (n+1) a_n x^{n+1} + \sum_{n=1}^{\infty} n a_{n-1} x^{n+1} \\
&= \sum_{n=0}^{\infty} \left[(n+1)a_n + n a_{n-1}\right] x^{n+1}, \qquad (a_{-1} := 0) \\
x^2 y_1'' &= x^2 \sum_{n=1}^{\infty} (n+1) n a_n x^{n-1} = \sum_{n=1}^{\infty} (n+1) n a_n x^{n+1} = \sum_{n=0}^{\infty} (n+1) n a_n x^{n+1}.
\end{aligned}$$

Substituting these into the ODE, we get

$$\begin{aligned}
&\sum_{n=0}^{\infty} (n+1) n a_n x^{n+1} - \sum_{n=0}^{\infty} \left[(n+1)a_n + n a_{n-1}\right] x^{n+1} + \sum_{n=0}^{\infty} a_n x^{n+1} \\
&= \sum_{n=0}^{\infty} (n^2 a_n - n a_{n-1}) x^{n+1} = 0.
\end{aligned}$$

$$\implies \quad a_n = \frac{a_{n-1}}{n} \quad \text{for} \quad n \geq 1$$

Setting $a_0 = 1$, we obtain $a_n = 1/n!$ and

$$y_1(x) = x \sum_{n=0}^{\infty} \frac{x^n}{n!} = x\, \mathrm{e}^x.$$

ii) For the determination of $y_2(x)$ we use the recurrence relation for $a_n(r)$ derived in the lecture; cf. also [BDM17], p. 223, Eq. (8). Since $F(r) = (r-1)^2$, $p_0 = p_1 = -1$, $q_0 = 1$ and all other coefficients p_i, q_i are zero, we have

$$\begin{aligned} a_n(r) &= -\frac{1}{F(r+n)} \sum_{k=0}^{n-1} \left[(r+k)p_{n-k} + q_{n-k}\right] a_k(r) \\ &= \frac{-1}{(r+n-1)^2}\,(r+n-1)p_1 a_{n-1}(r) = \frac{a_{n-1}(r)}{r+n-1} \qquad (n \geq 1). \end{aligned}$$

Setting $a_0(r) = 1$, we get

$$\begin{aligned} a_1(r) &= \frac{1}{r}, \\ a_2(r) &= \frac{1}{r(r+1)}, \\ &\vdots \\ a_n(r) &= \frac{1}{r(r+1)(r+2)\cdots(r+n-1)}. \end{aligned}$$

$$\begin{aligned} \Longrightarrow \quad b_n(r) &:= a_n'(r) = \frac{a_n'(r)}{a_n(r)}\,a_n(r) \\ &= -\left(\frac{1}{r} + \frac{1}{r+1} + \cdots + \frac{1}{r+n-1}\right) \frac{1}{r(r+1)(r+2)\cdots(r+n-1)} \end{aligned}$$

$$\Longrightarrow \quad b_n = b_n(1) = -\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right)\frac{1}{n!} = -\frac{\mathrm{H}_n}{n!}$$

$$\Longrightarrow \quad y_2(x) = \left(\sum_{n=0}^{\infty} \frac{x^{n+1}}{n!}\right) \ln x - \sum_{n=1}^{\infty} \frac{\mathrm{H}_n}{n!}\,x^{n+1} = x\,\mathrm{e}^x \ln x - \sum_{n=1}^{\infty} \frac{\mathrm{H}_n}{n!}\,x^{n+1}$$

The linear independency of $y_1(x)$, $y_2(x)$ was shown in the lecture.
A fundamental system of solutions on $(-\infty, 0)$ is formed by $y_1(x)$ and

$$y_2^-(x) = x\,\mathrm{e}^x \ln(-x) - \sum_{n=1}^{\infty} \frac{\mathrm{H}_n}{n!}\,x^{n+1}, \quad x \in (-\infty, 0).$$

Remark: The coefficients b_n can also be determined by substituting the „Ansatz“

for $y_2(x)$ into the ODE. Writing $L = x^2\mathrm{D}^2 - x(x+1)\mathrm{D} + \mathrm{id}$, we obtain

$$\begin{aligned}
y_2(x) &= y_1(x)\ln x + \sum_{n\geq 0} b_n x^n, \\
y_2'(x) &= y_1'(x)\ln x + \frac{y_1(x)}{x} + \sum_{n\geq 1} n b_n x^{n-1}, \\
y_2''(x) &= y_1''(x)\ln x + 2\,\frac{y_1(x)}{x} - \frac{y_1(x)}{x^2} + \sum_{n\geq 2} n(n-1) b_n x^{n-2}, \\
L\left[y_2(x)\right] &= L\left[y_1(x)\right]\ln x + 2x\,y_1'(x) - (x+2)y_1(x) + L\left[\sum_{n\geq 0} b_n x^n\right] \\
&= 0 + \underbrace{2x(x+1)\mathrm{e}^x - (x+2)x\mathrm{e}^x}_{=x^2\mathrm{e}^x} + \sum_{n=1}^{\infty}(n^2 b_n - n b_{n-1})x^{n+1} \\
&= \sum_{n=1}^{\infty}\left(n^2 b_n - n b_{n-1} + \frac{1}{(n-1)!}\right) x^{n+1}.
\end{aligned}$$

$L\left[y_2(x)\right] = 0$ is equivalent to an inhomogeneous linear recurrence relation for b_n, which has the particular solution $b_0 = 0$, $b_n = -\mathrm{H}_n/n!$ for $n \geq 1$ (as can be seen by introducing $B_n = n!b_n$, which satisfies $B_n - B_{n-1} = -1/n$).

c) Rewriting the ODE as

$$y'' + \frac{1}{x}y' + \left(\frac{1}{x^2} + \frac{1}{x}\right) y = 0,$$

we see that $x = 0$ is a regular singular point and

$$p_0 = \lim_{x\to 0} x\frac{x}{x^2} = 1, \quad q_0 = \lim_{x\to 0} x^2\frac{1+x}{x^2} = 1.$$

$\Longrightarrow$ The indicial equation is

$$r^2 + (p_0 - 1)r + q_0 = r^2 + 1 = 0.$$

$\Longrightarrow$ The exponents at the singularity $x = 0$ are $r_1 = \mathrm{i}$, $r_2 = -\mathrm{i}$. Thus there must be solutions $y_1(x)$, $y_2(x)$ on $(0,\infty)$ of the form

$$\begin{aligned}
y_1(x) &= x^{\mathrm{i}}\sum_{n=0}^{\infty} a_n x^n = \mathrm{e}^{\mathrm{i}\ln x}\sum_{n=0}^{\infty} a_n x^n, \\
y_2(x) &= x^{-\mathrm{i}}\sum_{n=0}^{\infty} a_n x^n = \mathrm{e}^{-\mathrm{i}\ln x}\sum_{n=0}^{\infty} a_n x^n.
\end{aligned}$$

This time we first determine the functions $a_n(r)$ from the recurrence relation and then substitute $r = \pm\mathrm{i}$. Since $p_1 = 0$, $q_1 = 1$, the recurrence relation for $a_n(r)$ is

$$a_n(r) = -\frac{a_{n-1}(r)}{F(r+n)} = -\frac{a_{n-1}(r)}{(r+n)^2 + 1}.$$

$$
\begin{aligned}
\Longrightarrow \quad a_1(r) &= -\frac{a_0(r)}{(r+1)^2+1} = -\frac{1}{(r+1)^2+1}, \\
a_2(r) &= \frac{1}{[(r+1)^2+1][(r+2)^2+1]}, \\
&\vdots \\
a_n(r) &= \frac{(-1)^n}{[(r+1)^2+1][(r+2)^2+1]\cdots[(r+n)^2+1]}, \\
\Longrightarrow y_1(x) &= \mathrm{e}^{\mathrm{i}\ln x}\left(1+\sum_{n=1}^{\infty}\frac{(-1)^n x^n}{[(1+\mathrm{i})^2+1][(2+\mathrm{i})^2+1]\cdots[(n+\mathrm{i})^2+1]}\right), \\
y_2(x) &= \mathrm{e}^{\mathrm{i}\ln x}\left(1+\sum_{n=1}^{\infty}\frac{(-1)^n x^n}{[(1-\mathrm{i})^2+1][(2-\mathrm{i})^2+1]\cdots[(n-\mathrm{i})^2+1]}\right).
\end{aligned}
$$

Two linearly independent real solutions $y_1^*(x)$, $y_2^*(x)$ are obtained by extracting real and imaginary part of $y_1(x)$, say.

$$
\begin{aligned}
y_1^*(x) &= \cos(\ln x)\left(1-\frac{x}{5}-\frac{x^2}{40}+\frac{3x^3}{520}\mp\cdots\right)-\sin(\ln x)\left(\frac{2x}{5}-\frac{3x^2}{40}+\frac{7x^3}{1560}\mp\cdots\right), \\
y_2^*(x) &= \sin(\ln x)\left(1-\frac{x}{5}-\frac{x^2}{40}+\frac{3x^3}{520}\mp\cdots\right)+\cos(\ln x)\left(\frac{2x}{5}-\frac{3x^2}{40}+\frac{7x^3}{1560}\mp\cdots\right).
\end{aligned}
$$

Differential Equations (Math 285)

H59 Do Exercises 5, 6, 9 in [BDM17], Ch. 5.7 (Exercises 6, 7, 10 in the 11th US edition). Optionally also show that $\mathrm{Y}_0'(x) = -\mathrm{Y}_1(x)$ for $x > 0$; see p. 236 (p. 238 in the 11th US edition) for the definition of $\mathrm{Y}_1(x)$. The solution $y_2(x)$ appearing in the definition of $\mathrm{Y}_1(x)$ is the same as that you obtain in Exercise 9 (resp., Exercise 10).

H60 The Γ function is defined for $x > 0$ by $\Gamma(x) = \int_0^\infty t^{x-1}\mathrm{e}^{-t}\,\mathrm{d}t$, and for non-integral $x < 0$ by choosing an integer $n > -x$ and setting

$$\Gamma(x) := \frac{\Gamma(x+n)}{x(x+1)\cdots(x+n-1)}.$$

a) Show that $\Gamma(x)$ is well-defined for $x < 0$, $x \notin \mathbb{Z}$, and satisfies $\Gamma(x+1) = x\,\Gamma(x)$ for all $x \in \mathbb{R} \setminus \{0, -1, -2, \dots\}$.
Hint: Recall from Calculus III that $\Gamma(x+1) = x\,\Gamma(x)$ for $x > 0$.

b) Show $\lim_{x\to -n} \frac{1}{\Gamma(x)} = 0$ for $n \in \mathbb{N} = \{0, 1, 2, \dots\}$.
This shows that $1/\Gamma$ can be continuously extended to $\mathbb{R}$ by defining $1/\Gamma(-n) := 0$ for $n \in \mathbb{N}$.

c) The Bessel function of order $\nu \in \mathbb{R}$ is defined as (cf. the lecture)

$$\mathrm{J}_\nu(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{2^{\nu+2m} m!\,\Gamma(\nu+m+1)}\, x^{\nu+2m} \quad \text{for } x \in \mathbb{R},$$

cf. b) for the definition of $1/\Gamma(\nu+m+1)$.
Show $\mathrm{J}_{-\nu} = (-1)^\nu \mathrm{J}_\nu$ for $\nu \in \mathbb{N}$.
Hint: Show first that the coefficients of x^n in the expansion of $\mathrm{J}_{-\nu}(x)$ are zero for $n < \nu$.

H61 Find the Laplace transforms of

a) $1 + 2t + 3t^2$; b) e^{5t+3}; c) $\int_0^t \tau \sin\tau \,\mathrm{d}\tau$; d) $\sin^3 t$.

H62 Find inverse Laplace transforms of

a) $\dfrac{5}{s+6}$; b) $\dfrac{2s-1}{s^2+3}$; c) $\dfrac{1}{(s^2+1)(s^2+4)}$; d) $\dfrac{\mathrm{d}}{\mathrm{d}s}\dfrac{1-\mathrm{e}^{-5s}}{s}$;

e) $\ln\dfrac{s}{s-1}$; f) $\ln\dfrac{s^2+1}{(s-1)^2}$; g) $\dfrac{s+1}{s^2(s^2+1)}$; h) $\dfrac{\mathrm{e}^{-2s}-\mathrm{e}^{-4s}}{s}$;

i) $\operatorname{arccot}\dfrac{s}{\omega}$; j) $\dfrac{s^2-1}{(s^3+s^2-5s+3)(s^2-4)}$.

Six answers suffice.

H63 Solve the following initial value problems with the Laplace transform (two answers suffice):

a) $y'' - 3\,y' + 2\,y = 6\,\mathrm{e}^{-t}, \quad y(0) = 9,\ y'(0) = 6;$

b) $y'' + 2\,y' - 3\,y = 6\sinh(2t), \quad y(0) = 0,\ y'(0) = 4;$

c) $y''' + y'' - 5\,y' + 3\,y = 6\sinh(2t), \quad y(0) = y'(0) = 0,\ y''(0) = 4.$

H64 Find the Laplace transform of the Bessel function J_0 in one of two ways (the other is optional):

a) From the power series of J_0 and termwise integration of the Laplace integral.
Hint: The power series expansion

$$\frac{1}{\sqrt{1-4x}} = \sum_{n=0}^{\infty} \binom{2n}{n} x^n, \quad \text{valid for } |x| < 1/4,$$

may help (but you should prove it first).

b) From the Bessel ODE of order $\nu = 0$.

H65 *Optional Exercise*

For $x \in \mathbb{R} \setminus \{0\}$, $\nu \in \mathbb{R}$ show:

a) $\mathrm{J}_{\nu+1}(x) = \frac{2\nu}{x}\,\mathrm{J}_\nu(x) - \mathrm{J}_{\nu-1}(x);$

b) $\mathrm{J}_\nu'(x) = -\mathrm{J}_{\nu+1}(x) + \frac{\nu}{x}\,\mathrm{J}_\nu(x).$

Remark: a) Provides a recurrence relation to determine J_ν for $\nu \in \mathbb{N}$ from J_0, J_1. The Neumann functions Y_ν, $\nu \in \mathbb{N}$, satisfy the same recurrence relation and provide a 2nd solution of $x^2y'' + xy' + (x^2 - \nu^2)y = 0$, which is linearly independent of J_ν. Thus in order to determine Y_ν for $\nu \in \mathbb{N}$ (the only case of interest) it suffices to know Y_0 and Y_1.

H66 *Optional Exercise*

Suppose $F(s) = \mathcal{L}\{f(t)\}$ is defined for $\mathrm{Re}(s) > a$, $a \in [-\infty, \infty)$. Show that $\lim_{s\to+\infty} F(s) = 0$; cp. Exercise 24 in [BDM17], Ch. 6.1.
Hint: Use the uniform convergence of $\int_0^\infty f(t)\mathrm{e}^{-st}$ on $\mathrm{Re}(s) \geq a + 1$ (resp., for $a = -\infty$ on $\mathrm{Re}(s) \geq 0$).

Due on Thu May 16, 10 am

Using the Laplace transform as a tool for the solution of certain IVP's (required for H63) will be discussed in the lecture on Sat May 11.

H65 and H66 can be handed in until Mon May 20, 10 am.

Solutions (prepared by Li Menglu and TH)

59 a) Exercise 5

Using the ratio test, we get

$$\begin{aligned}
\lim_{m\to+\infty}\left|\frac{a_{m+1}}{a_m}\right| &= \lim_{m\to+\infty}\left|\frac{\frac{(-1)^{m+1}x^{2(m+1)}}{2^{2(m+1)}((m+1)!)^2}}{\frac{(-1)^m x^{2m}}{2^{2m}(m!)^2}}\right| \\
&= \lim_{m\to+\infty}\left|\frac{-x^2}{4(m+1)^2}\right| \\
&= \lim_{m\to+\infty}\frac{x^2}{4(m+1)^2} \\
&= 0 \\
&< 1
\end{aligned}$$

for all $x \neq 0$.

So, the series for $\mathrm{J}_0(x)$ converges absolutely for all x.

b) Exercise 6

Using the ratio test, we get

$$\begin{aligned}
\lim_{m\to+\infty}\left|\frac{a_{m+1}}{a_m}\right| &= \lim_{m\to+\infty}\left|\frac{\frac{(-1)^{m+1}x^{2(m+1)}}{2^{2(m+1)}(m+2)!(m+1)!}}{\frac{(-1)^m x^{2m}}{2^{2m}(m+1)!m!}}\right| \\
&= \lim_{m\to+\infty}\left|\frac{-x^2}{4(m+2)(m+1)}\right| \\
&= \lim_{m\to+\infty}\frac{x^2}{4(m+2)(m+1)} \\
&= 0 \\
&< 1
\end{aligned}$$

for all $x \neq 0$.

So, the series for $\mathrm{J}_1(x)$ converges absolutely for all x.

It follows that we can obtain the derivative of $\mathrm{J}_0(x)$ everywhere by term-wise differentiation:

$$\begin{aligned}
\mathrm{J}_0'(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\sum_{m=0}^{\infty}\frac{(-1)^m x^{2m}}{2^{2m}(m!)^2} \\
&= \sum_{m=1}^{\infty}\frac{(-1)^m x^{2m-1}}{2^{2m-1}m!(m-1)!} \\
&= \sum_{m=0}^{\infty}\frac{(-1)^{m+1} x^{2m+1}}{2^{2m+1}(m+1)!m!} \\
&= -\frac{x}{2}\sum_{m=0}^{\infty}\frac{(-1)^m x^{2m}}{2^{2m}(m+1)!m!} \\
&= -\mathrm{J}_1(x),
\end{aligned}$$

as claimed.

c) Exercise 9

First, we want to show that $a_1(-1) = a_1'(-1) = 0$.

Equation (24) in Ch. 5.7 gives

$$a_1(r)((r+1)^2 - 1)x^{r+1} = 0.$$

Hence $a_1(r) = 0$ for $r \notin \{-2, 0\}$, and in particular $a_1(-1) = a_1'(-1) = 0$. (Alternatively, look at the recurrence relation $a_n(r) = -F(r+n)^{-1}\sum_{k=0}^{n-1}\left[(r+k)p_{n-k} + q_{n-k}\right]a_k(r)$, which for $n = 1$ reduces to $a_1(r) = -\frac{1}{r(r+2)}\left[rp_1 + q_1\right]a_0(r) = \frac{0}{r(r+2)}$, since for the Bessel equation $p_1 = q_1 = 0$.) Next,

$$c_1(-1) = \frac{\mathrm{d}}{\mathrm{d}r}[(r+1)a_1(r)]\bigg|_{r=-1} = 0.$$

Then, from equation (25) in Ch. 5.7 or using the said general recurrence relation for $a_n(r)$, we get

$$a_n(r) = \frac{-a_{n-2}(r)}{(r+n-1)(r+n+1)} \qquad \text{for } n \geq 2.$$

Since $a_1(r) = 0$, this gives $a_n(r) = 0$ for all odd n wherever $a_n(r)$ is defined (i,e., $r \notin \{0, -2, -4, \ldots, -n-1\}$), and hence $c_n(-1) = \frac{\mathrm{d}}{\mathrm{d}r}\left[(r+1)a_n(r)\right]\big|_{r=-1} = 0$ for all odd n. For even n the recurrence relation gives by induction

$$\begin{aligned}
a_2(r) &= \frac{-a_0(r)}{(r+1)(r+3)} = -\frac{1}{(r+1)(r+3)},\\
a_4(r) &= \frac{-a_2(r)}{(r+3)(r+5)} = \frac{1}{(r+1)(r+3)(r+3)(r+5)},\\
&\vdots\\
a_{2m}(r) &= \frac{-1}{(r+2m-1)(r+2m+1)} \cdot \frac{-1}{(r+2m-3)(r+2m-1)} \cdots \frac{-1}{(r+1)(r+3)}\\
&= \frac{(-1)^m}{(r+1)(r+3)\cdots(r+2m-1)(r+3)(r+5)\cdots(r+2m+1)}.
\end{aligned}$$

So,

$$\begin{aligned}
c_{2m}(-1) &= \frac{\mathrm{d}}{\mathrm{d}r}[(r+1)a_{2m}(r)]\bigg|_{r=-1}\\
&= \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{(-1)^m}{(r+3)^2(r+5)^2\cdots(r+2m-1)^2(r+2m+1)}\right)\bigg|_{r=-1}\\
&= \left[\left(-\frac{2}{r+3} - \frac{2}{r+5} - \frac{2}{r+2m-1} - \frac{1}{r+2m+1}\right)(r+1)a_{2m}(r)\right]\bigg|_{r=-1}\\
&= \left(-1 - \frac{1}{2} - \cdots - \frac{1}{m-1} - \frac{1}{2m}\right)\frac{(-1)^m}{2^2 4^2 \cdots (2m-2)^2(2m)}\\
&= -\frac{1}{2}(\mathrm{H}_{m-1} + \mathrm{H}_m)\frac{(-1)^m}{2^{2m-1}m!(m-1)!}\\
&= \frac{(-1)^{m+1}(\mathrm{H}_{m-1} + \mathrm{H}_m)}{2^{2m}m!(m-1)!} \qquad \text{for } m = 1, 2, \ldots
\end{aligned}$$

Finally, we need to compute

$$\begin{aligned} a &= \lim_{r\to -1}(r+1)a_2(r) \\ &= \lim_{r\to -1}\left(\frac{-1}{r+3}\right) \\ &= -\frac{1}{2}. \end{aligned}$$

According to the theory (e.g., Th. 5.6.1 in Ch. 5.6), a 2nd solution of Bessel's equation of order one is

$$\begin{aligned} y_2(x) &= -\frac{1}{2}y_1(x)\ln|x| + \frac{1}{|x|}\left(1 - \sum_{m=1}^{\infty}\frac{(-1)^m(\mathrm{H}_m+\mathrm{H}_{m-1})}{2^{2m}m!(m-1)!}x^{2m}\right) \\ ` &= -\mathrm{J}_1(x)\ln|x| + \frac{1}{|x|}\left(1 - \sum_{m=1}^{\infty}\frac{(-1)^m(\mathrm{H}_m+\mathrm{H}_{m-1})}{2^{2m}m!(m-1)!}x^{2m}\right), \quad x\neq 0. \end{aligned}$$

For this note that $y_1(x)$ denotes the analytic solution normalized by $a_0 = y_1'(0) = 1$, so that $\mathrm{J}_1(x) = \sum_{m=0}^{\infty}\frac{(-1)^m}{2^{2m+1}m!(m+1)!}x^{2m+1} = \frac{x}{2}+\cdots = \frac{1}{2}y_1(x)$.
The corresponding Neumann function is then

$$\begin{aligned} \mathrm{Y}_1(x) &= \frac{2}{\pi}\left[-y_2(x) + (\gamma - \ln 2)\mathrm{J}_1(x)\right] \\ &= \frac{2}{\pi}\left[\left(\ln\frac{x}{2}+\gamma\right)\mathrm{J}_1(x) - \frac{1}{x} + \sum_{m=1}^{\infty}\frac{(-1)^m(\mathrm{H}_m+\mathrm{H}_{m-1})}{2^{2m}m!(m-1)!}x^{2m-1}\right]. \end{aligned}$$

Finally we show that $\mathrm{Y}_0'(x) = -\mathrm{Y}_1(x)$.

$$\begin{aligned} \mathrm{Y}_0'(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\frac{2}{\pi}\left[\left(\ln\frac{x}{2}+\gamma\right)\mathrm{J}_0(x) + \sum_{m=1}^{\infty}\frac{(-1)^{m+1}\mathrm{H}_m}{2^{2m}(m!)^2}x^{2m}\right] \\ &= \frac{2}{\pi}\left[\frac{\mathrm{J}_0(x)}{x} + \left(\ln\frac{x}{2}+\gamma\right)\mathrm{J}_0'(x) + \sum_{m=1}^{\infty}\frac{(-1)^{m+1}\mathrm{H}_m}{2^{2m-1}m!(m-1)!}x^{2m-1}\right] \\ &= \frac{2}{\pi}\left[-\left(\ln\frac{x}{2}+\gamma\right)\mathrm{J}_1(x) + \frac{1}{x} + \sum_{m=1}^{\infty}\left(\frac{(-1)^m}{2^{2m}(m!)^2} + \frac{(-1)^{m+1}\mathrm{H}_m}{2^{2m-1}m!(m-1)!}\right)x^{2m-1}\right] \\ &= \frac{2}{\pi}\left[-\left(\ln\frac{x}{2}+\gamma\right)\mathrm{J}_1(x) + \frac{1}{x} - \sum_{m=1}^{\infty}\frac{\frac{(-1)^{m-1}}{m} + (-1)^m 2\mathrm{H}_m}{2^{2m}m!(m-1)!}x^{2m-1}\right] \\ &= -\mathrm{Y}_1(x), \end{aligned}$$

since $2\mathrm{H}_m - \frac{1}{m} = \mathrm{H}_m + \mathrm{H}_{m-1}$.

60 Before we solve the exercise, we show that Γ satisfies the indicated identity for $x > 0$ and any positive integer n. This follows by repeated application of the functional equation

$\Gamma(x+1) = x\,\Gamma(x)$, which was shown in Calculus III:

$$\begin{aligned}
\Gamma(x) &= \frac{\Gamma(x+1)}{x} && (\text{since } \Gamma(x+1) = x\,\Gamma(x)) \\
&= \frac{\Gamma(x+2)}{x(x+1)} && (\text{since } \Gamma(x+2) = (x+1)\Gamma(x+1)) \\
&= \frac{\Gamma(x+3)}{x(x+1)(x+2)} && (\text{since } \Gamma(x+3) = (x+2)\Gamma(x+2)) \\
&= \cdots = \frac{\Gamma(x+n)}{x(x+1)\cdots(x+n-1)}.
\end{aligned}$$

This identity suggest how to extend the definition of Γ to negative x, because its right-hand side makes sense for $x > -n$, $x \notin \{0, 1, \ldots, n-1\}$.

a) To show that $\Gamma(x)$ is well-defined for $x < 0$, $x \notin \mathbb{Z}$, we need to show that different choices of $n > -x$ don't affect the value of $\Gamma(x)$ as specified in the exercise. The smallest n we can use in the definition is $n = \lceil -x \rceil$. With this n we have

$$\begin{aligned}
\Gamma(x) &= \frac{\Gamma(x+n)}{x(x+1)\cdots(x+n-1)} \\
&= \frac{\Gamma(x+n+1)}{x(x+1)\cdots(x+n-1)(x+n)} && (\text{since } \Gamma(x+n+1) = (x+n)\Gamma(x+n)) \\
&= \frac{\Gamma(x+n+2)}{x(x+1)\cdots(x+n-1)(x+n)(x+n+1)} && (\text{same reasoning}) \\
&= \cdots
\end{aligned}$$

Thus using $n+1$, $n+2$, … in the formula to compute $\Gamma(x)$ yields the same result as n, which means that $\Gamma(x)$ is well-defined.
Then, we prove that $\Gamma(x+1) = x\,\Gamma(x)$. For $x > 0$ this was shown in Calculus III, so it remains to consider the case $x < 0$, $x \notin \mathbb{Z}$. Choose $n \in \mathbb{N}$ such that $x + n > 0$. Then in the definition of $\Gamma(x+1)$ we can use $n-1$, since $x + 1 + (n-1) = x + n > 0$.

$$\begin{aligned}
\Longrightarrow \Gamma(x+1) &= \frac{\Gamma(x+1+n-1)}{(x+1)(x+2)\cdots(x+1+(n-1)-1)} \\
&= \frac{\Gamma(x+n)}{(x+1)(x+2)\cdots(x+n-1)} \\
&= x\,\frac{\Gamma(x+n)}{x(x+1)(x+2)\cdots(x+n-1)} \\
&= x\,\Gamma(x)
\end{aligned}$$

For $n = 1$, which is possible only if $-1 < x < 0$, the definition of $\Gamma(x)$ reduces to $\Gamma(x) = \frac{\Gamma(x+1)}{x}$ and the functional equation holds as well. This case is included in the above computation, provided the first denominator is interpreted as 1 (empty product).

b) For x close to $-n$ we have $x + n + 1 > 0$. Hence a) gives

$$\lim_{x\to -n} \frac{1}{\Gamma(x)} = \lim_{x\to -n} \frac{x(x+1)\cdots(x+n)}{\Gamma(x+n+1)}.$$

Since $\Gamma(1) = 1$, the limit evaluates to

$$\lim_{x \to -n} \frac{1}{\Gamma(x)} = \frac{(-n)(-n+1)\cdots(0)}{1} = 0.$$

Thus $1/\Gamma$ has zeros at the nonpositive integers, and Γ has poles there; cf. Fig. 1. (It is easy to see that $\Gamma(x) \approx \frac{(-1)^n}{n!(x+n)}$ for $x \to -n$ and hence $1/\Gamma(x) \approx (-1)^n n!(x+n)$ for $x \to -n$. In particular, the poles of Γ and the zeros of $1/\Gamma$ are simple.)

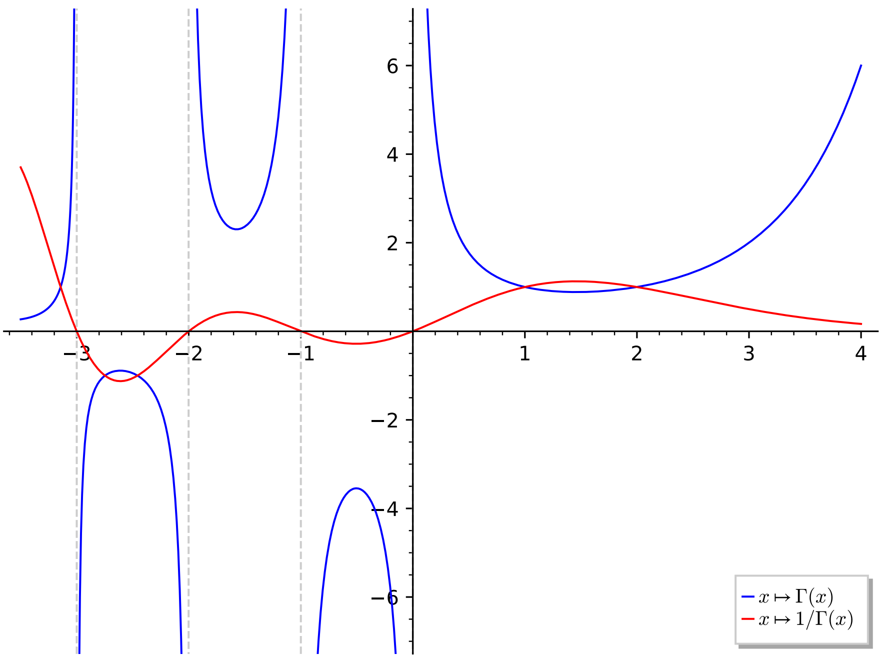


Figure 1: The Γ-function on $\mathbb{R}$ and its reciprocal function $1/\Gamma$

Remark: The Euler integral $\Gamma(z) = \int_0^\infty t^{z-1} \mathrm{e}^{-t} \,\mathrm{d}t$ makes sense for $z \in \mathbb{C}$ with $\operatorname{Re} z > 0$ and gives an extension of Γ to the open right half plane H of $\mathbb{C}$ satisfying $\Gamma(z+1) = z\,\Gamma(z)$ for all $z \in H$. Using the same idea as above, one can extend the domain of Γ further to $\mathbb{C} \setminus \{0, -1, -2, \dots\}$ (preserving the functional equation) and that of $1/\Gamma$ to the whole complex plane. The resulting functions are analytic in their domain. Thus, in a way, the "disconnected picture" of Γ on the negative real axis is misleading: In the complex plane there are only the isolated poles at $z = 0, -1, -2, \dots$

c) First, we have

$$\mathrm{J}_{-\nu}(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{2^{-\nu+2m} m!\, \Gamma(-\nu+m+1)}\, x^{-\nu+2m}.$$

From b), we know that $1/\Gamma(-n) = 0$ for $n \in \mathbb{N}$. So, the coefficients of $x^{-\nu+2m}$ are zero for $m < \nu$. Then

$$\begin{aligned}
\mathrm{J}_{-\nu}(x) &= \sum_{m=\nu}^{\infty} \frac{(-1)^m}{2^{-\nu+2m} m!\,\Gamma(-\nu+m+1)}\, x^{-\nu+2m} \\
&= \sum_{m=\nu}^{\infty} \frac{(-1)^m}{2^{\nu+2(m-\nu)} m!\,\Gamma((m-\nu)+1)}\, x^{\nu+2(m-\nu)} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^{n+\nu}}{2^{\nu+2n} (n+\nu)!\,\Gamma(n+1)}\, x^{\nu+2n} \qquad \text{(let } n = m-\nu) \\
&= (-1)^{\nu} \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{\nu+2n} (n+\nu)! n!}\, x^{\nu+2n} \\
&= (-1)^{\nu} \mathrm{J}_{\nu}(x).
\end{aligned}$$

61 a) $\mathcal{L}\{1+2t+3t^2\} = \mathcal{L}\{1\} + 2\,\mathcal{L}\{t\} + 3\,\mathcal{L}\{t^2\} = 1/s + 2/s^2 + 6/s^3$ for $\mathrm{Re}(s) > 0$;

b) $\mathcal{L}\{\mathrm{e}^{5t+3}\} = \mathrm{e}^3 \mathcal{L}\{\mathrm{e}^{5t}\} = \mathrm{e}^3/(s-5)$ for $\mathrm{Re}(s) > 5$;

c) $\mathcal{L}\left\{\int_0^t \tau \sin\tau \,\mathrm{d}\tau\right\} = \frac{1}{s}\,\mathcal{L}\{t\sin t\} = -\frac{1}{s}\frac{\mathrm{d}}{\mathrm{d}s}\,\mathcal{L}\{\sin t\} = -\frac{1}{s}\frac{\mathrm{d}}{\mathrm{d}s}\frac{1}{s^2+1} = -\frac{1}{s}\frac{-2s}{(s^2+1)^2} = \frac{2}{(s^2+1)^2}$. Alternatively (but more costly), evaluate the integral first using integration by parts, $\int_0^t \tau \sin\tau\,\mathrm{d}\tau = \sin t - t\cos t$, and then recall $\frac{1}{(s^2+1)^2} = \mathcal{L}\left\{\frac{1}{2}(\sin t - t\cos t)\right\}$ from the lecture.

d) From $\sin(3t) = \mathrm{Im}(\cos t + \mathrm{i}\sin t)^3 = 3\cos^2 t \sin t - \sin^3 t = 3\sin t - 4\sin^3 t$ we get $\mathcal{L}\left\{\sin^3 t\right\} = \mathcal{L}\left\{\frac{1}{4}(3\sin t - \sin(3t))\right\} = \frac{1}{4}\left(\frac{3}{s^2+1} - \frac{3}{s^2+9}\right) = \frac{6}{(s^2+1)(s^2+9)}$.

62 a) $\mathcal{L}^{-1}\left\{\frac{5}{s+6}\right\} = 5\,\mathcal{L}^{-1}\left\{\frac{1}{s+6}\right\} = 5\,\mathrm{e}^{-6t}$;

b) $\mathcal{L}^{-1}\left\{\frac{2s-1}{s^2+3}\right\} = 2\,\mathcal{L}^{-1}\left\{\frac{s}{s^2+3}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s^2+3}\right\} = 2\cos(\sqrt{3}\,t) - \frac{1}{\sqrt{3}}\sin(\sqrt{3}\,t)$;

c) $\frac{1}{(s^2+1)(s^2+4)} = \frac{1}{3}\left(\frac{1}{s^2+1} - \frac{1}{s^2+4}\right) \Longrightarrow \mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)(s^2+4)}\right\} = \frac{1}{3}\left(\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\}\right) = \frac{1}{3}\sin t - \frac{1}{6}\sin(2t)$;

d) $\frac{1-\mathrm{e}^{-5s}}{s} = \mathcal{L}\{H(t) - H(t-5)\} \Longrightarrow \frac{\mathrm{d}}{\mathrm{d}s}\frac{1-\mathrm{e}^{-5s}}{s} = \mathcal{L}\{-t\,H(t) + t\,H(t-5)\}$, i.e., $\mathcal{L}^{-1}\left\{\frac{\mathrm{d}}{\mathrm{d}s}\frac{1-\mathrm{e}^{-5s}}{s}\right\} = -t\,H(t) + t\,H(t-5)$;

e) We have

$$\ln\frac{s}{s-1} = \ln\frac{1}{1-1/s} = -\ln(1-1/s) = \frac{1}{s} + \frac{1}{2s^2} + \frac{1}{3s^3} + \frac{1}{4s^4} + \cdots$$

for $|s| > 1$, and hence

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\ln\frac{s}{s-1}\right\} &= 1 + \frac{t}{2} + \frac{t^2}{3\,2!} + \frac{t^3}{4\,3!} + \cdots = \sum_{n=1}^{\infty} \frac{t^{n-1}}{n!} \\
&= \frac{\mathrm{e}^t - 1}{t}.
\end{aligned}$$

f) Let $F(s) = \ln \frac{s^2+1}{(s-1)^2} = \ln(s^2+1) - 2\ln(s-1)$ and $f(t) = \mathcal{L}^{-1}\{F(s)\}$.

$$\begin{aligned}
\implies \quad & \mathcal{L}\{-t\,f(t)\} = F'(s) = \frac{2s}{s^2+1} - \frac{2}{s-1} = \mathcal{L}\left\{2\cos t - 2\,\mathrm{e}^t\right\} \\
\implies \quad & -t\,f(t) = 2\cos t - 2\,\mathrm{e}^t \\
\implies \quad & f(t) = \frac{2\,\mathrm{e}^t - 2\cos t}{t} \qquad (t \geq 0)
\end{aligned}$$

Since $\mathrm{e}^0 = \cos 0 = 1$ this is in fact an everywhere analytic function of t.

g) We have

$$\frac{s+1}{s^2(s^2+1)} = \frac{1}{s(s^2+1)} + \frac{1}{s^2(s^2+1)} = \frac{1}{s} - \frac{s}{s^2+1} + \frac{1}{s^2} - \frac{1}{s^2+1}$$

$$\implies \mathcal{L}\left\{\frac{s+1}{s^2(s^2+1)}\right\} = 1 - \cos t + t - \sin t.$$

h) $\mathcal{L}^{-1}\left\{\left(\mathrm{e}^{-2s} - \mathrm{e}^{-4s}\right)/s\right\} = \mathcal{L}^{-1}\left\{\mathrm{e}^{-2s}/s\right\} - \mathcal{L}^{-1}\left\{\mathrm{e}^{-4s}/s\right\} = H(t-2) - H(t-4)$.

i) From the lecture we know $\mathcal{L}\left\{\frac{\sin t}{t}\right\} = \operatorname{arccot} s$. Dilation in the domain gives

$$\mathcal{L}\left\{\frac{\sin(\omega t)}{\omega t}\right\} = \frac{1}{\omega}\operatorname{arccot}\frac{s}{\omega}. \quad \implies \quad \mathcal{L}^{-1}\left\{\operatorname{arccot}\frac{s}{\omega}\right\} = \frac{\sin(\omega t)}{t}$$

j) We have

$$\frac{s^2-1}{s^3+s^2-5s+3} = \frac{s+1}{(s-1)(s+3)} = \frac{1}{2}\left(\frac{1}{s-1} + \frac{1}{s+3}\right).$$

$$\implies \mathcal{L}^{-1}\left\{\frac{s^2-1}{s^3+s^2-5s+3}\right\} = \frac{1}{2}\left(\mathrm{e}^t + \mathrm{e}^{-3t}\right).$$

63 As usual, we denote the Laplace transform of $y(t)$ by $Y(s)$

a) Applying $\mathcal{L}$ to both sides of the equation and inserting the initial conditions gives

$$\begin{aligned}
s^2\,Y(s) - 9s - 6 - 3\big(s\,Y(s) - 9\big) + 2\,Y(s) &= \frac{6}{s+1} \\
(s^2 - 3s + 2)Y(s) &= \frac{6}{s+1} + 9s - 21 = \frac{9s^2 - 12s - 15}{s+1} \\
Y(s) &= \frac{9s^2 - 12s - 15}{(s-1)(s-2)(s+1)}
\end{aligned}$$

The partial fraction decomposition of $Y(s)$ is

$$Y(s) = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{s+1}$$

with

$$\begin{aligned}
A &= (s-1)Y(s)\big|s = 1 = 9, \\
A &= (s-2)Y(s)\big|s = 2 = -1, \\
C &= (s+1)Y(s)\big|s = -1 = 1, \\
\implies Y(s) &= \frac{9}{s-1} - \frac{1}{s-2} + \frac{1}{s+1} \\
\implies y(t) &= \mathcal{L}^{-1}\{y(s)\} = 9\,\mathrm{e}^t - \mathrm{e}^{2t} + \mathrm{e}^{-t}.
\end{aligned}$$

b) The Laplace transform of $\sinh t = \frac{1}{2}(\mathrm{e}^t - \mathrm{e}^{-t})$ is $F(s) = \frac{1}{2}\left(\frac{1}{s-1} - \frac{1}{s+1}\right) = \frac{1}{s^2-1}$, from which $\mathcal{L}\{\sinh(2t)\} = \frac{1}{2}F\left(\frac{s}{2}\right) = \frac{1/2}{(s/2)^2-1} = \frac{2}{s^2-4}$.

$$\Longrightarrow s^2\,Y(s) - 4 + 2s\,Y(s) - 3\,Y(s) = \frac{12}{s^2-4}$$

$$(s^2+2s-3)Y(s) = \frac{12}{s^2-4} + 4 = \frac{4\,s^2-4}{s^2-4}$$

$$Y(s) = \frac{4(s^2-1)}{(s^2+2s-3)(s^2-4)} = \frac{4(s+1)}{(s+3)(s-2)(s+2)}$$

The partial fraction decomposition of $Y(s)$ is

$$Y(s) = \frac{A}{s+3} + \frac{B}{s-2} + \frac{C}{s+2}$$

with

$$A = (s+3)Y(s)\big|s = -3 = -8/5,$$
$$A = (s-2)Y(s)\big|s = 2 = 3/5,$$
$$C = (s+2)Y(s)\big|s = -2 = 1,$$
$$\Longrightarrow Y(s) = -\frac{8/5}{s+3} + \frac{3/5}{s-2} + \frac{1}{s+2}$$
$$\Longrightarrow y(t) = -\frac{8}{5}\,\mathrm{e}^{-3t} + \frac{3}{5}\,\mathrm{e}^{2t} + \mathrm{e}^{-2t}.$$

c)

$$s^3\,Y(s) - 4 + s^2\,Y(s) - 5s\,Y(s) + 3\,Y(s) = \frac{12}{s^2-4}$$

$$Y(s) = \frac{4(s^2-1)}{(s^3+s^2-5s+3)(s^2-4)} = \frac{4(s+1)}{(s-1)(s+3)(s-2)(s+2)}$$

The partial fraction decomposition of $Y(s)$ is (details omitted)

$$Y(s) = \frac{2}{5\,(s+3)} - \frac{1}{3\,(s+2)} - \frac{2}{3\,(s-1)} + \frac{3}{5\,(s-2)}.$$
$$\Longrightarrow y(t) = \frac{2}{5}\,\mathrm{e}^{-3t} - \frac{1}{3}\,\mathrm{e}^{-2t} - \frac{2}{3}\,\mathrm{e}^{t} + \frac{3}{5}\,\mathrm{e}^{2t}.$$

64 a) We have

$$\mathrm{J}_0(t) = \sum_{m=0}^{\infty} \frac{(-1)^m}{4^m (m!)^2}\,t^{2m} = \sum_{m=0}^{\infty} \frac{(-1)^m}{4^m}\binom{2m}{m}\frac{t^{2m}}{(2m)!}.$$

$$\Longrightarrow \mathcal{L}\{\mathrm{J}_0\} = \sum_{m=0}^{\infty} \frac{(-1)^m}{4^m}\binom{2m}{m}\frac{1}{s^{2m+1}} = \frac{1}{s}\sum_{m=0}^{\infty}\binom{2m}{m}\left(-\frac{1}{4s^2}\right)^m$$
$$= \frac{1}{s}\frac{1}{\sqrt{1-4\left(-\frac{1}{4s^2}\right)}} \qquad \text{(using the hint)}$$
$$= \frac{1}{\sqrt{s^2+1}}.$$

The computation is valid for $|s| > 1$, since the binomial series involved (see below) has radius of convergence 1; cf. the theorem about termwise integration of Laplace integrals in the lecture.

Finally we prove the asserted series expansion:

$$\begin{aligned}\binom{-1/2}{m} &= \frac{-\frac{1}{2}\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\cdots\left(-\frac{2m-1}{2}\right)}{m!} = (-1)^m\frac{1\cdot 3\cdot 5\cdots(2m-1)}{m!\,2^m}\\ &= (-1)^m\frac{(2m)!}{m!\,2^m\cdot 2\cdot 4\cdot 6\cdots(2m)} = (-1)^m\frac{(2m)!}{(m!)^2 4^m} = \frac{(-1)^m}{4^m}\binom{2m}{m},\end{aligned}$$

and therefore

$$\sum_{m=0}^{\infty}\binom{2m}{m}x^m = \sum_{m=0}^{\infty}(-1)^m 4^m\binom{-1/2}{m}x^m = \sum_{m=0}^{\infty}\binom{-1/2}{m}(-4x)^m = (1-4x)^{-1/2},$$

using the binomial series.

J_0 is the solution of the IVP $t\,y'' + y' + t\,y = 0$, $y(0) = 1$, $y'(0) = 0$. Writing $Y(s) = \mathcal{L}\{\mathrm{J}_0(t)\}$ and taking the Laplace transform on both sides gives

$$\begin{aligned}-\frac{\mathrm{d}}{\mathrm{d}s}\left(s^2\,Y(s) - s\right) + s\,Y(s) - 1 - Y'(s) &= 0\\ -\left(s^2\,Y'(s) + 2s\,Y(s) - 1\right) + s\,Y(s) - 1 - Y'(s) &= 0\\ Y'(s) &= -\frac{s}{s^2+1}\,Y(s)\\ \Longrightarrow\ Y(s) = c\exp\int_0^s -\frac{1}{2}\ln(\sigma^2+1)\mathrm{d}\sigma &= \frac{c}{\sqrt{s^2+1}} \quad \text{for some constant } c.\end{aligned}$$

The constant c can be determined from

$$\mathcal{L}\{\mathrm{J}_0'(t)\} = s\,Y(s) - \mathrm{J}_0(0)$$

and the general fact that Laplace transforms tend to zero for $s\to\infty$. It follows that

$$c = \lim_{s\to\infty}\frac{cs}{\sqrt{s^2+1}} = \lim_{s\to\infty} s\,Y(s) = \mathrm{J}_0(0) = 1,$$

and hence $\mathcal{L}\{\mathrm{J}_0(t)\} = Y(s) = 1/\sqrt{s^2+1}$.

65 For $\nu\in\mathbb{N}$ the function $\mathrm{J}_\nu(x)$ was defined in the lecture as the analytic solution of Bessel's equation of order ν normalized by setting the coefficient of x^ν (first nonzero coefficient) equal to $\frac{1}{2^\nu \nu!}$. It can also be derived using Frobenius' method as follows (not part of the exercise):

$$\begin{aligned}0 &= x^2y'' + xy' + (x^2-\nu^2)y\\ &= \sum_{n=0}^{\infty}a_n(r+n)(r+n-1)x^{r+n} + \sum_{n=0}^{\infty}a_n(r+n)x^{r+n} + (x^2-\nu^2)\cdot\sum_{n=0}^{\infty}a_n x^{r+n}\\ &= \sum_{n=0}^{\infty}a_n[(r+n)^2-\nu^2]x^{r+n} + \sum_{n=2}^{\infty}a_{n-2}x^{r+n}\\ &= a_0(r^2-\nu^2)x^r + a_1[(r+1)^2-\nu^2]x^{r+1} + \sum_{n=2}^{\infty}\left\{[(r+n)^2-\nu^2]a_n + a_{n-2}\right\}x^{r+n}\end{aligned}$$

For $r = \nu$ there are solutions with arbitrary a_0. These must satisfy $a_n = 0$ for all odd n and $[(\nu+n)^2 - \nu^2]a_n + a_{n-2} = n(n+2\nu)a_n + a_{n-2}0$ for all even $n \geq 2$. By induction,

$$\begin{aligned} a_{2m} &= -\frac{a_{2m-2}}{2m(2m+2\nu)} = \cdots = \frac{(-1)^m a_0}{[2m(2m-2)\cdots 2]\,[(2m+2\nu)(2m-2+2\nu)\cdots(2+2\nu)]} \\ &= \frac{(-1)^m a_0}{2^{2m} m!(\nu+1)(\nu+2)\cdots(\nu+m)}. \end{aligned}$$

Choosing $a_0 = \frac{1}{2^\nu \nu!}$, we get

$$\mathrm{J}_\nu(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{\nu+2m}}{2^{\nu+2m} m!(\nu+m)!}.$$

Then, we solve the exercise:

a)

$$\begin{aligned} \frac{2\nu}{x}\mathrm{J}_\nu(x) - \mathrm{J}_{\nu-1}(x) &= \sum_{m=0}^{\infty} \frac{(-1)^m x^{\nu+2m-1}\nu}{2^{\nu+2m-1} m!(\nu+m)!} - \sum_{m=0}^{\infty} \frac{(-1)^m x^{\nu+2m-1}}{2^{\nu+2m-1} m!(\nu+m-1)!} \\ &= \sum_{m=0}^{\infty} \frac{(-1)^m x^{\nu+2m-1}}{2^{\nu+2m-1} m!(\nu+m-1)!} \left(\frac{\nu}{\nu+m} - 1 \right) \\ &= \sum_{m=1}^{\infty} \frac{(-1)^{m+1} x^{\nu+2m-1}}{2^{\nu+2m-1} (m-1)!(\nu+m)!} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^{n+2} x^{\nu+2n+1}}{2^{\nu+2n+1} n!(\nu+n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{\nu+1+2n}}{2^{\nu+1+2n} n!(\nu+1+n)!} \\ &= \mathrm{J}_{\nu+1}(x) \end{aligned}$$

b) The Bessel functions may be differentiated termwise to yield

$$\begin{aligned} \mathrm{J}_\nu'(x) &= \frac{\mathrm{d}}{\mathrm{d}x} \sum_{m=0}^{\infty} \frac{(-1)^m}{2^{\nu+2m}\, m!\, \Gamma(m+\nu+1)}\, x^{\nu+2m} \\ &= \sum_{m=0}^{\infty} \frac{(-1)^m (\nu+2m)}{2^{\nu+2m}\, m!\, \Gamma(m+\nu+1)}\, x^{\nu+2m-1} \\ &= \sum_{m=0}^{\infty} \frac{(-1)^m \nu}{2^{\nu+2m} m!\, \Gamma(m+\nu+1)}\, x^{\nu+2m-1} + \sum_{m=1}^{\infty} \frac{(-1)^m}{2^{\nu+2m-1}(m-1)!\, \Gamma(m+\nu+1)}\, x^{\nu+2m-1} \\ &= \frac{\nu}{x}\, \mathrm{J}_\nu(x) + \sum_{m=0}^{\infty} \frac{(-1)^{m+1}}{2^{\nu+2m+1} m!\, \Gamma(m+\nu+2)}\, x^{\nu+2m+1} \\ &= \frac{\nu}{x}\, \mathrm{J}_\nu(x) - \mathrm{J}_{\nu+1}(x). \end{aligned}$$

66 The assertion "$\lim_{s\to+\infty} F(s) = 0$", which tacitly assumes $s \in \mathbb{R}$, can in fact be strenghtened to $\lim_{\mathrm{Re}(s)\to+\infty} F(s) = 0$, as the subsequent proof shows. But the complex limit $\lim_{|s|\to\infty} F(s)$ need not exist, because near the line of convergence $F(s)$ may be unbounded.

Let $\epsilon > 0$ be given. Since the Laplace integral converges uniformly for $\operatorname{Re} s \geq a + 1$ ($\operatorname{Re} s \geq 0$), as shown in the lecture, we can find $R > 0$ such that $\left|\int_R^\infty f(t)\mathrm{e}^{-st}\,\mathrm{d}t\right| < \epsilon/2$ for all such s. Assuming that f is piecewise continuous, hence bounded on $[0, R]$, there exists $M > 0$ such $|f(t)| \leq M$ for $t \in [0, R]$. Writing $s = x + \mathrm{i}y$, we then have

$$\left|\int_0^R f(t)\mathrm{e}^{-st}\,\mathrm{d}t\right| \leq \int_0^R |f(t)|\,\mathrm{e}^{-xt}\,\mathrm{d}t \leq M\int_0^R \mathrm{e}^{-xt}\,\mathrm{d}t = \frac{M(1-\mathrm{e}^{-xR})}{x} \leq \frac{M}{x},$$

provided that $x > 0$. For $x > 2M/\epsilon$ the right-hand side is $\epsilon/2$.

$$\begin{aligned} \implies |F(s)| &= \left|\int_0^R f(t)\mathrm{e}^{-st}\,\mathrm{d}t + \int_R^\infty f(t)\mathrm{e}^{-st}\,\mathrm{d}t\right| \\ &\leq \left|\int_0^R f(t)\mathrm{e}^{-st}\,\mathrm{d}t\right| + \left|\int_R^\infty f(t)\mathrm{e}^{-st}\,\mathrm{d}t\right| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \end{aligned}$$

provided that $\operatorname{Re}(s) > \max\{a+1, 0, 2M/\epsilon\}$. This shows $\lim_{\operatorname{Re}(s)\to+\infty} F(s) = 0$.

Differential Equations (Math 285)

H67 Solve the following IVP's with the Laplace transform:

a) $y'' + y' + y = \mathrm{u}_\pi(t) - \mathrm{u}_{2\pi}(t), \quad y(0) = 1,\ y'(0) = 0;$

b) $y'' + 2y' + y = \begin{cases} \sin(2t) & \text{if } 0 \leq t \leq \pi/2, \\ 0 & \text{if } t > \pi/2, \end{cases} \quad y(0) = 1,\ y'(0) = 0.$

H68 Do Exercises 11 and 20 in [BDM17], Ch. 7.1.

H69 For the matrix

$$\mathbf{A} = \begin{pmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{pmatrix}$$

solve the initial value problem $\mathbf{y}' = \mathbf{A}\mathbf{y}$, $\mathbf{y}(0) = (0, 1, 0)^\mathsf{T}$, and determine $\lim_{t\to+\infty} \mathbf{y}(t)$ for the solution.

Hint: The solution to Exercise H43 of Homework 9 in Math257 (Fall 2023) may help.

H70 Determine a fundamental system of solutions of $\mathbf{y}' = \mathbf{B}\mathbf{y}$ for the matrix

$$\mathbf{B} = \begin{pmatrix} 0 & 1 & -2 & -1 & 2 \\ 5 & -2 & -3 & -2 & 3 \\ 14 & 3 & -12 & -5 & 9 \\ 13 & 3 & -8 & -8 & 8 \\ 16 & 3 & -10 & -6 & 7 \end{pmatrix}$$

Hint: Use the result of Exercise H60 of Homework 12 in Math257 (Fall 2023).

H71 Consider again the matrix $\mathbf{A}$ from H69. Determine the matrix exponential function $\mathrm{e}^{\mathbf{A}t}$ in two ways,

a) using the fundamental matrix $\mathbf{\Phi}(t)$ obtained in H69 and the formula $\mathrm{e}^{\mathbf{A}t} = \mathbf{\Phi}(t)\mathbf{\Phi}(0)^{-1}$;

b) using the "new method" for computing $\mathrm{e}^{\mathbf{A}t}$ discussed in Lecture 38 (tentatively); cf. also H48 of Homework 9.

H72 *Optional Exercise*

a) Suppose $\mathbf{A} \in \mathbb{R}^{n\times n}$ has n distinct eigenvalues $\lambda_1, \dots, \lambda_n$. Show that

$$\mathrm{e}^{\mathbf{A}t} = \sum_{i=1}^{n} \mathrm{e}^{\lambda_i t} \ell_i(\mathbf{A}),$$

where $\ell_i(X) = \prod_{j=1, j\neq i}^{n} \frac{X-\lambda_j}{\lambda_i-\lambda_j}$ are the corresponding Lagrange polynomials.

b) Suppose that $\mathbf{A} \in \mathbb{R}^{n \times n}$ is symmetric and $\mathbf{v}_1, \dots, \mathbf{v}_n$ is an orthonormal basis of $\mathbb{R}^n$ consisting of eigenvectors of $\mathbf{A}$. Show that

$$\mathrm{e}^{\mathbf{A}t} = \sum_{i=1}^{n} \mathrm{e}^{\lambda_i t} \mathbf{v}_i \mathbf{v}_i^{\mathsf{T}},$$

where λ_i is the eigenvalue corresponding to $\mathbf{v}_i$. (Note that the vectors $\mathbf{v}_i$ are column vectors, and hence $\mathbf{v}_i \mathbf{v}_i^{\mathsf{T}}$ are $n \times n$ matrices of rank 1.)

c) The matrix considered in H69 and H71 satisfies both conditions. Use a) and b) to give two further evaluations of its matrix exponential function.

Due on Mon May 20, 10 am

The material on linear systems required for solving H70, H71, H72 will be discussed in the lectures on Thu May 16 and Fri May 17. The optional Exercise H72 should also be handed in on Mon May 20.

Solutions (prepared by Zhang Zhuhaobo, Niu Yiqun, and TH)

67 a) The transformed ODE is

$$s^2\,Y(s) - s + s\,Y(s) - 1 + Y(s) = \frac{\mathrm{e}^{-\pi s} - \mathrm{e}^{-2\pi s}}{s}$$

$$Y(s) = \frac{\mathrm{e}^{-\pi s} - \mathrm{e}^{-2\pi s}}{s(s^2+s+1)} + \frac{s+1}{s^2+s+1} = \frac{\mathrm{e}^{-\pi s} - \mathrm{e}^{-2\pi s}}{s} + \left(1 - \mathrm{e}^{-\pi s} + \mathrm{e}^{2\pi s}\right)\frac{s+1}{s^2+s+1}$$

using $\frac{1}{s(s^2+s+1)} = \frac{1}{s} - \frac{s+1}{s^2+s+1}$. The first summand has inverse Laplace transform $\mathrm{u}_\pi(t) - \mathrm{u}_{2\pi}(t)$. The second summand can be rewritten as

$$\left(1 - \mathrm{e}^{-\pi s} + \mathrm{e}^{2\pi s}\right)\frac{s + \frac{1}{2} + \frac{1}{2}}{\left(s+\frac{1}{2}\right)^2 + \frac{3}{4}}$$

and has inverse Laplace transform $y_1(t) - \mathrm{u}_\pi(t)y_1(t-\pi) + \mathrm{u}_{2\pi}(t)y_1(t-2\pi)$ with

$$y_1(t) = \mathrm{e}^{-t/2}\cos\frac{\sqrt{3}\,t}{2} + \frac{1}{\sqrt{3}}\,\mathrm{e}^{-t/2}\sin\frac{\sqrt{3}\,t}{2}.$$

$\Longrightarrow$ The solution is

$$\begin{aligned} y(t) &= y_1(t) + \mathrm{u}_\pi(t)\bigl(1 - y_1(t-\pi)\bigr) + \mathrm{u}_{2\pi(t)}\bigl(y_1(t-2\pi) - 1\bigr) \\ &= \begin{cases} y_1(t) & \text{if } 0 \le t \le \pi, \\ 1 + y_1(t) - y_1(t-\pi) & \text{if } \pi \le t \le 2\pi, \\ y_1(t) - y_1(t-\pi) + y_1(t-2\pi) & \text{if } t \ge 2\pi. \end{cases} \end{aligned}$$

Remark: The solution on $[0,\pi]$, viz. $y_1(t)$, is the solution of the IVP $y'' + y' + y = 0$,

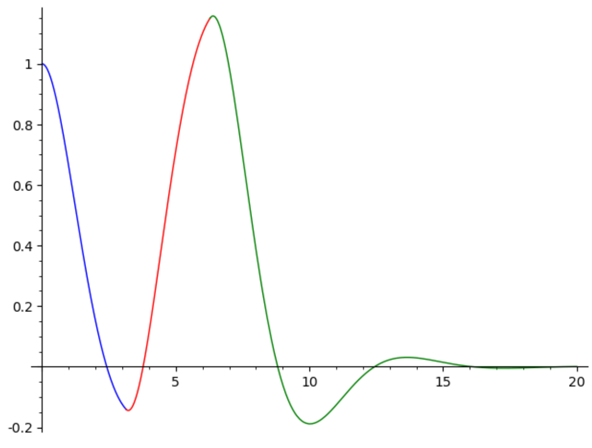


Figure 1: The solution $y(t)$ of H67 a)

$y(0) = 1$, $y'(0) = 0$, as can also be checked using our earlier discussion of higher order linear ODE's with constant coefficients. The solution on $[\pi, 2\pi]$ is obtained by adding to $y_1(t)$ the solution of the IVP $y'' + y' + y = 1$, $y(\pi) = y'(\pi) = 0$ (to fit the initial conditions at $t = \pi$), which is $y_2(t) = 1 - y_1(t-\pi)$. The solution on $[2\pi, \infty)$ is obtained by adding to this in turn the solution of the IVP $y'' + y' + y = -1$, $y(2\pi) = y'(2\pi) = 0$ (to fit the initial conditions at $t = 2\pi$), which is $y_3(t) = y_1(t-2\pi) - 1$.

b) Here the forcing function is $\sin(2t) - \mathrm{u}_{\pi/2}(t)\sin(2t) = \sin(2t) + \mathrm{u}_{\pi/2}(t)\sin\bigl(2(t-\pi/2)\bigr)$ and the transformed ODE is

$$
\begin{aligned}
s^2\,Y(s) - s + 2\bigl(s\,Y(s) - 1\bigr) + Y(s) &= \frac{2}{s^2+4}\left(1 + \mathrm{e}^{-\pi s/2}\right)\\
Y(s) &= \frac{2 + 2\,\mathrm{e}^{-\pi s/2}}{(s^2+4)(s+1)^2} + \frac{s+2}{(s+1)^2}
\end{aligned}
$$

The real partial fractions decomposition of $\frac{1}{(s^2+4)(s+1)^2}$ is

$$
\begin{aligned}
\frac{1}{(s^2+4)(s+1)^2} &= -\frac{2\,s+3}{25\,(s^2+4)} + \frac{2}{25\,(s+1)} + \frac{1}{5\,(s+1)^2}.\\
\implies Y(s) &= \left(2 + 2\,\mathrm{e}^{-\pi s/2}\right)\left(-\frac{2\,s+3}{25\,(s^2+4)} + \frac{2}{25\,(s+1)} + \frac{1}{5\,(s+1)^2}\right) + \frac{1}{s+1} + \frac{1}{(s+1)^2}\\
\implies y(t) &= -\frac{4}{25}\cos(2t) - \frac{3}{25}\sin(2t) + \frac{4}{25}\mathrm{e}^{-t} + \frac{2}{5}t\mathrm{e}^{-t}\\
&\quad - \frac{4}{25}\mathrm{u}_{\pi/2}(t)\cos(2t-\pi) - \frac{3}{25}\mathrm{u}_{\pi/2}(t)\sin(2t-\pi) + \frac{4}{25}\mathrm{u}_{\pi/2}(t)\mathrm{e}^{-(t-\pi/2)}\\
&\quad + \frac{2}{5}\mathrm{u}_{\pi/2}(t)\,(t-\pi/2)\,\mathrm{e}^{-(t-\pi/2)}\\
&\quad + \mathrm{e}^{-t} + t\mathrm{e}^{-t}\\
&= -\frac{4}{25}\cos(2t) - \frac{3}{25}\sin(2t) + \frac{29}{25}\mathrm{e}^{-t} + \frac{7}{5}t\mathrm{e}^{-t}\\
&\quad + \frac{4}{25}\mathrm{u}_{\pi/2}(t)\cos(2t) + \frac{3}{25}\mathrm{u}_{\pi/2}(t)\sin(2t) + \frac{(4-5\pi)\mathrm{e}^{\pi/2}}{25}\mathrm{u}_{\pi/2}(t)\mathrm{e}^{-t} + \frac{2\mathrm{e}^{\pi/2}}{5}\mathrm{u}_{\pi/2}(t)t\,\mathrm{e}^{-t}\\
&= \begin{cases} -\frac{4}{25}\cos(2t) - \frac{3}{25}\sin(2t) + \frac{29}{25}\,\mathrm{e}^{-t} + \frac{7}{5}\,t\,\mathrm{e}^{-t} & \text{if } t \le \pi/2,\\ \frac{29+(4-5\pi)\mathrm{e}^{\pi/2}}{25}\,\mathrm{e}^{-t} + \frac{7+2\mathrm{e}^{\pi/2}}{5}\,t\,\mathrm{e}^{-t} & \text{if } t \ge \pi/2. \end{cases}
\end{aligned}
$$

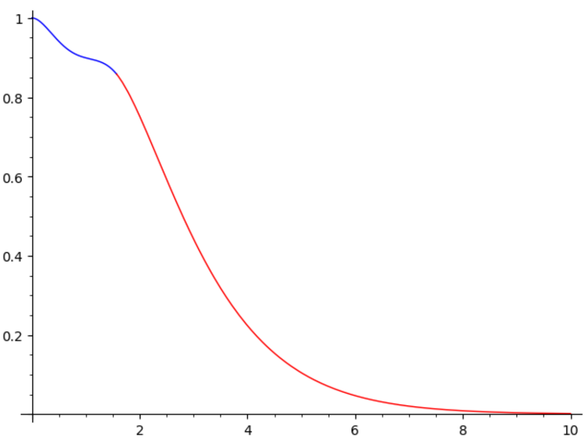


Figure 2: The solution $y(t)$ of H67 b)

68 Ex. 11 From Figure (b) we can easily get the forces acting on each block. Assume the positive direction of the motion is to the right.

For Block 1: $a = \frac{d^2x_1}{dt^2}$,

$$\begin{aligned} m_1\frac{d^2x_1}{dt^2} &= m_1a = -k_1x_1 + F_1(t) + k_2(x_2 - x_1) \\ &= -(k_1+k_2)x_1 + k_2x_2 + F_1(t) \end{aligned}$$

Similarly, for Block 2: $a = \frac{d^2x_2}{dt^2}$,

$$\begin{aligned} m_2\frac{d^2x_2}{dt^2} &= m_2a = -k_2(x_2 - x_1) + F_2(t) - k_3x_2 \\ &= k_2x_1 - (k_2+k_3)x_2 + F_2(t) \end{aligned}$$

Ex. 20 According to the current-voltage relation for each element in the circuit, we have

$$\begin{aligned} V_L &= L\frac{\mathrm{d}I}{\mathrm{d}t}, \\ I_C &= C\frac{\mathrm{d}V}{\mathrm{d}t}, \\ V_1 &= I_1R_1, \\ V_2 &= I_2R_2. \end{aligned}$$

Applying Kirchhoff's voltage law to the L-R_1-R_2 loop gives

$$L\frac{\mathrm{d}I}{\mathrm{d}t} + I_1R_1 + I_2R_2 = 0.$$

But $I_1 = I$, $I_2R_2 = V_2 = V$, and hence $L\frac{\mathrm{d}I}{\mathrm{d}t} + R_1I + V = 0$, proving the first equation. Applying Kirchhoff's current law to the lower node gives

$$I_1 = I_2 + I_C = \frac{V_2}{R_2} + C\frac{\mathrm{d}V}{\mathrm{d}t}.$$

Since $I_1 = I$, $V_2 = V$, the second equation follows.

69 Eigenvalues/eigenvectors of $\mathbf{A}$ were determined in Exercise H43. The following computation is copied from there.
The eigenvalues are the roots of the characteristic equation.

$$\begin{aligned} \det|\mathbf{A} - \lambda\mathbf{I}| &= \begin{vmatrix} -1-\lambda & 1 & 0 \\ 1 & -2-\lambda & 1 \\ 0 & 1 & -1-\lambda \end{vmatrix} \\ &= -(\lambda+1)^2(\lambda+2) + 2(\lambda+1) = -(\lambda+1)(\lambda^2+3\lambda) = -\lambda(\lambda+1)(\lambda+3) = 0. \end{aligned}$$

Thus the eigenvalues are $\lambda_1 = 0$, $\lambda_2 = -1$ and $\lambda_3 = -3$. To find the eigenvectors, we replace λ by each of the eigenvalues in turn.
For $\lambda_1 = 0$,

$$\begin{pmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

$$\mathbf{x}^{(1)} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

For $\lambda_2 = -1$,

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

$$\mathbf{x}^{(2)} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

For $\lambda_3 = -3$,

$$\begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

$$\mathbf{x}^{(3)} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}.$$

Therefore, the ODE system $\mathbf{y}' = \mathbf{A}\mathbf{y}$ has the general solution

$$y(t) = c_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + c_3 e^{-3t} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}.$$

Then consider the initial value problem $\mathbf{y}' = \mathbf{A}\mathbf{y}$, $\mathbf{y}(0) = (0, 1, 0)^\mathsf{T}$:

$$y(0) = \begin{pmatrix} c_1 + c_2 + c_3 \\ c_1 - 2c_3 \\ c_1 - c_2 + c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\Longrightarrow c_1 = \frac{1}{3},\ c_2 = 0,\ c_3 = -\frac{1}{3}$$

Therefore, the solution of the initial value problem is

$$y(t) = \begin{pmatrix} \frac{1}{3} - \frac{1}{3}e^{-3t} \\ \frac{1}{3} + \frac{2}{3}e^{-3t} \\ \frac{1}{3} - \frac{1}{3}e^{-3t} \end{pmatrix}.$$

Moreover,

$$\lim_{t \to +\infty} \mathbf{y}(t) = \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix}$$

70 In Exercise H60 it was shown that

$$\mathbf{S} = \begin{pmatrix} 0 & 1 & 1 & 1 & -2 \\ 1 & 1 & 0 & -5 & 0 \\ 0 & 3 & 2 & 0 & -1 \\ 0 & 3 & 1 & 0 & -2 \\ 0 & 3 & 1 & 0 & 1 \end{pmatrix},$$

transforms **B** into Jordan canonical form; more precisely,

$$\mathbf{S}^{-1}\mathbf{B}\mathbf{S} = \mathbf{J} = \left(\begin{array}{ccc|cc} -3 & 0 & 0 & 0 & 0 \\ 1 & -3 & 0 & 0 & 0 \\ 0 & 1 & -3 & 0 & 0 \\ \hline 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 1 & -3 \end{array}\right).$$

Denoting the columns of **S**, in order, by $\mathbf{v}_1, \ldots, \mathbf{v}_5$ (different from the numbering used in the solution of H60!), this says

$$\begin{aligned} \mathbf{B}\mathbf{v}_1 &= -3\mathbf{v}_1 + \mathbf{v}_2, & (\mathbf{B} + 3\,\mathbf{I})\,\mathbf{v}_1 &= \mathbf{v}_2, \\ \mathbf{B}\mathbf{v}_2 &= -3\mathbf{v}_2 + \mathbf{v}_3, & (\mathbf{B} + 3\,\mathbf{I})\,\mathbf{v}_2 &= \mathbf{v}_3, \\ \mathbf{B}\mathbf{v}_3 &= -3\mathbf{v}_3, & (\mathbf{B} + 3\,\mathbf{I})\,\mathbf{v}_3 &= \mathbf{0}, \\ \mathbf{B}\mathbf{v}_4 &= -3\mathbf{v}_4 + \mathbf{v}_5, & (\mathbf{B} + 3\,\mathbf{I})\,\mathbf{v}_4 &= \mathbf{v}_5, \\ \mathbf{B}\mathbf{v}_1 &= -3\mathbf{v}_5, & (\mathbf{B} + 3\,\mathbf{I})\,\mathbf{v}_5 &= \mathbf{0}. \end{aligned}$$

According to the general theory (use the two chains $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ and $\mathbf{v}_4, \mathbf{v}_5$), a fundamental system of solutions of $\mathbf{y}' = \mathbf{B}\mathbf{y}$ is then

$$\begin{aligned} \mathbf{y}_1(t) &= \mathrm{e}^{-3t}\mathbf{v}_1 + t\,\mathrm{e}^{-3t}\mathbf{v}_2 + \tfrac{1}{2}\,t^2\mathrm{e}^{-3t}\mathbf{v}_3 \\ &= \mathrm{e}^{-3t}\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + t\,\mathrm{e}^{-3t}\begin{pmatrix} 1 \\ 1 \\ 3 \\ 3 \\ 3 \end{pmatrix} + \tfrac{1}{2}t^2\mathrm{e}^{-3t}\begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \\ 1 \end{pmatrix} = \mathrm{e}^{-3t}\begin{pmatrix} t + \frac{1}{2}\,t^2 \\ 1 + t \\ 3t + t^2 \\ 3t + \frac{1}{2}\,t^2 \\ 3t + \frac{1}{2}\,t^2 \end{pmatrix}, \end{aligned}$$

$$\mathbf{y}_2(t) = \mathrm{e}^{-3t}\mathbf{v}_2 + t\,\mathrm{e}^{-3t}\mathbf{v}_3 = \mathrm{e}^{-3t}\begin{pmatrix} 1 \\ 1 \\ 3 \\ 3 \\ 3 \end{pmatrix} + t\,\mathrm{e}^{-3t}\begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \\ 1 \end{pmatrix} = \mathrm{e}^{-3t}\begin{pmatrix} 1 + t \\ 1 \\ 3 + 2t \\ 3 + t \\ 3 + t \end{pmatrix},$$

$$\mathbf{y}_3(t) = \mathrm{e}^{-3t}\mathbf{v}_3 = \mathrm{e}^{-3t}\begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \\ 1 \end{pmatrix},$$

$$\mathbf{y}_4(t) = \mathrm{e}^{-3t}\mathbf{v}_4 + t\,\mathrm{e}^{-3t}\mathbf{v}_5 = \mathrm{e}^{-3t}\begin{pmatrix} 1 \\ -5 \\ 0 \\ 0 \\ 0 \end{pmatrix} + t\,\mathrm{e}^{-3t}\begin{pmatrix} -2 \\ 0 \\ -1 \\ -2 \\ 1 \end{pmatrix} = \mathrm{e}^{-3t}\begin{pmatrix} 1 - 2t \\ -5 \\ -t \\ -2t \\ t \end{pmatrix},$$

$$\mathbf{y}_5(t) = \mathrm{e}^{-3t}\mathbf{v}_5 = \mathrm{e}^{-3t}\begin{pmatrix} -2 \\ 0 \\ -1 \\ -2 \\ 1 \end{pmatrix}.$$

71 a) From the solution of H69 we have

$$\mathbf{\Phi}(t)=\begin{pmatrix}1 & e^{-t} & e^{-3t}\\ 1 & 0 & -2e^{-3t}\\ 1 & -e^{-t} & e^{-3t}\end{pmatrix},\quad \mathbf{\Phi}(0)=\begin{pmatrix}1 & 1 & 1\\ 1 & 0 & -2\\ 1 & -1 & 1\end{pmatrix},\quad \mathbf{\Phi}(0)^{-1}=\begin{pmatrix}\frac{1}{3} & \frac{1}{3} & \frac{1}{3}\\ \frac{1}{2} & 0 & -\frac{1}{2}\\ \frac{1}{6} & -\frac{1}{3} & \frac{1}{6}\end{pmatrix}.$$

$$\begin{aligned}\Longrightarrow e^{\mathbf{A}t}&=\begin{pmatrix}1 & e^{-t} & e^{-3t}\\ 1 & 0 & -2e^{-3t}\\ 1 & -e^{-t} & e^{-3t}\end{pmatrix}\begin{pmatrix}\frac{1}{3} & \frac{1}{3} & \frac{1}{3}\\ \frac{1}{2} & 0 & -\frac{1}{2}\\ \frac{1}{6} & -\frac{1}{3} & \frac{1}{6}\end{pmatrix}\\ &=\begin{pmatrix}\frac{1}{3}+\frac{1}{2}e^{-t}+\frac{1}{6}e^{-3t} & \frac{1}{3}-\frac{1}{3}e^{-3t} & \frac{1}{3}-\frac{1}{2}e^{-t}+\frac{1}{6}e^{-3t}\\ \frac{1}{3}-\frac{1}{3}e^{-3t} & \frac{1}{3}+\frac{2}{3}e^{-3t} & \frac{1}{3}-\frac{1}{3}e^{-3t}\\ \frac{1}{3}-\frac{1}{2}e^{-t}+\frac{1}{6}e^{-3t} & \frac{1}{3}-\frac{1}{3}e^{-3t} & \frac{1}{3}+\frac{1}{2}e^{-t}+\frac{1}{6}e^{-3t}\end{pmatrix}\end{aligned}$$

b) We apply the new method with $a(X)=\chi_{\mathbf{A}}(X)=X(X+1)(X+3)$; cf. the solution of H69. A fundamental system of solutions of $\chi_{\mathbf{A}}(\mathrm{D})y=0$ is 1, e^{-t}, e^{-3t}, which has Wronski matrix

$$\mathbf{W}(t)=\begin{pmatrix}1 & e^{-t} & e^{-3t}\\ 0 & -e^{-t} & -3\,e^{-3t}\\ 0 & e^{-t} & 9\,e^{-3t}\end{pmatrix}.$$

The special fundamental system $c_0(t)$, $c_1(t)$, $c_2(t)$ required for the computation of $e^{\mathbf{A}t}$ is obtained by multiplying the first row of $\mathbf{W}(t)$ with $\mathbf{W}(0)^{-1}$. The standard method for matrix inversion gives

$$\left[\begin{array}{ccc|ccc}1 & 1 & 1 & 1 & 0 & 0\\ 0 & -1 & -3 & 0 & 1 & 0\\ 0 & 1 & 9 & 0 & 0 & 1\end{array}\right]\to\left[\begin{array}{ccc|ccc}1 & 0 & -2 & 1 & 1 & 0\\ 0 & 1 & 3 & 0 & -1 & 0\\ 0 & 0 & 6 & 0 & 1 & 1\end{array}\right]\to\left[\begin{array}{ccc|ccc}1 & 0 & 0 & 1 & \frac{4}{3} & \frac{1}{3}\\ 0 & 1 & 0 & 0 & -\frac{3}{2} & -\frac{1}{2}\\ 0 & 0 & 1 & 0 & \frac{1}{6} & \frac{1}{6}\end{array}\right].$$

$$\begin{aligned}\Longrightarrow \big(c_0(t),c_1(t),c_2(t)\big)&=\big(1,e^{-t},e^{-3t}\big)\begin{pmatrix}1 & \frac{4}{3} & \frac{1}{3}\\ 0 & -\frac{3}{2} & -\frac{1}{2}\\ 0 & \frac{1}{6} & \frac{1}{6}\end{pmatrix}\\ &=\big(1,\tfrac{4}{3}-\tfrac{3}{2}e^{-t}+\tfrac{1}{6}e^{-3t},\tfrac{1}{3}-\tfrac{1}{2}e^{-t}+\tfrac{1}{6}e^{-3t}\big)\end{aligned}$$

$$\begin{aligned}\Longrightarrow\ e^{\mathbf{A}t}&=c_0(t)\mathbf{I}_3+c_1(t)\mathbf{A}+c_2(t)\mathbf{A}^2\\ &=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}+\big(\tfrac{4}{3}-\tfrac{3}{2}e^{-t}+\tfrac{1}{6}e^{-3t}\big)\begin{pmatrix}-1 & 1 & 0\\ 1 & -2 & 1\\ 0 & 1 & -1\end{pmatrix}+\big(\tfrac{1}{3}-\tfrac{1}{2}e^{-t}+\tfrac{1}{6}e^{-3t}\big)\begin{pmatrix}2 & -3 & 1\\ -3 & 6 & -3\\ 1 & -3 & 2\end{pmatrix}\\ &=\begin{pmatrix}\frac{1}{3}+\frac{1}{2}e^{-t}+\frac{1}{6}e^{-3t} & \frac{1}{3}-\frac{1}{3}e^{-3t} & \frac{1}{3}-\frac{1}{2}e^{-t}+\frac{1}{6}e^{-3t}\\ \frac{1}{3}-\frac{1}{3}e^{-3t} & \frac{1}{3}+\frac{2}{3}e^{-3t} & \frac{1}{3}-\frac{1}{3}e^{-3t}\\ \frac{1}{3}-\frac{1}{2}e^{-t}+\frac{1}{6}e^{-3t} & \frac{1}{3}-\frac{1}{3}e^{-3t} & \frac{1}{3}+\frac{1}{2}e^{-t}+\frac{1}{6}e^{-3t}\end{pmatrix}.\end{aligned}$$

The computation is facilitated by the symmetry properties of $\mathbf{I}_3$, $\mathbf{A}$, $\mathbf{A}^2$, which are all of the form $\left(\begin{smallmatrix}a & b & c\\ b & d & b\\ c & b & a\end{smallmatrix}\right)$, leaving only 4 entries of $e^{\mathbf{A}t}$ to be determined.

72 a) We show that $\mathbf{\Phi}(t)=\sum_{i=1}^n \mathrm{e}^{\lambda_i t}\ell_i(\mathbf{A})$ solves the matrix IVP $\mathbf{\Phi}'(t)=\mathbf{A}\mathbf{\Phi}(t)$, $\mathbf{\Phi}(0)=\mathbf{I}_n$. Since $t\mapsto \mathrm{e}^{\mathbf{A}t}$ is the unique solution of this IVP, this proves the assertion.

We have

$$\mathbf{\Phi}'(t)=\sum_{i=1}^n \lambda_i \mathrm{e}^{\lambda_i t}\ell_i(\mathbf{A}),$$
$$\mathbf{A}\mathbf{\Phi}(t)=\sum_{i=1}^n \mathrm{e}^{\lambda_i t}\mathbf{A}\ell_i(\mathbf{A}).$$

By the Cayley-Hamilton Theorem,

$$\mathbf{0}=\chi_{\mathbf{A}}(\mathbf{A})=(\mathbf{A}-\lambda_1\mathbf{I})\cdots(\mathbf{A}-\lambda_n\mathbf{I})=c(\mathbf{A}-\lambda_i\mathbf{I})\ell_i(\mathbf{A}),$$

where $c=\prod_{j=1,j\neq i}^n(\lambda_i-\lambda_j)\neq 0$. Hence $(\mathbf{A}-\lambda_i\mathbf{I})\ell_i(\mathbf{A})=\mathbf{0}$ and $\mathbf{A}\ell_i(\mathbf{A})=\lambda_i\ell_i(\mathbf{A})$. This proves $\mathbf{\Phi}'(t)=\mathbf{A}\mathbf{\Phi}(t)$.

Further, we have

$$\mathbf{\Phi}(0)=\sum_{i=1}^n \ell_i(\mathbf{A}).$$

The Lagrange polynomials satisfy the identity $\sum_{i=1}^n \ell_i(X)=1$ (since $p(X)=\sum_{i=1}^n \ell_i(X)$ has degree $<n$ and solves the interpolation problem $p(\lambda_i)=1$, $1\leq i\leq n$). Substituting $\mathbf{A}$ into this identity gives $\sum_{i=1}^n \ell_i(\mathbf{A})=\mathbf{I}_n$, as desired.

b) From the lecture we know that $\mathrm{e}^{\mathbf{A}t}$ is the matrix with eigenvectors $\mathbf{v}_i$ and corresponding eigenvalues $\mathrm{e}^{\lambda_i t}$. Clearly this determines $\mathrm{e}^{\mathbf{A}t}$ uniquely. Setting $\mathbf{\Phi}(t)=\sum_{i=1}^n \mathrm{e}^{\lambda_i t}\mathbf{v}_i\mathbf{v}_i^{\mathsf{T}}$, we have

$$\mathbf{\Phi}(t)\mathbf{v}_j=\sum_{i=1}^n \mathrm{e}^{\lambda_i t}\mathbf{v}_i\mathbf{v}_i^{\mathsf{T}}\mathbf{v}_j=\mathrm{e}^{\lambda_j t}\mathbf{v}_j,$$

since $\mathbf{v}_i^{\mathsf{T}}\mathbf{v}_j=\mathbf{v}_i\cdot\mathbf{v}_j=0$ for $i\neq j$ and $\mathbf{v}_j^{\mathsf{T}}\mathbf{v}_j=1$. This proves the assertion.

c) Firstly, we use the Lagrange polynomials. From the solution of H69 we have $\lambda_1=0$, $\lambda_2=-1$, $\lambda_3=-3$. The corresponding Lagrange polynomials are

$$\ell_1(X)=\tfrac{1}{3}(X+1)(X+3),$$
$$\ell_2(X)=-\tfrac{1}{2}X(X+3),$$
$$\ell_3(X)=\tfrac{1}{6}X(X+1).$$

$$(\mathbf{A}+\mathbf{I})(\mathbf{A}+3\mathbf{I})=\begin{pmatrix}0&1&0\\1&-1&1\\0&1&0\end{pmatrix}\begin{pmatrix}2&1&0\\1&1&1\\0&1&2\end{pmatrix}=\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix},$$
$$\mathbf{A}(\mathbf{A}+3\mathbf{I})=\begin{pmatrix}-1&1&0\\1&-2&1\\0&1&-1\end{pmatrix}\begin{pmatrix}2&1&0\\1&1&1\\0&1&2\end{pmatrix}=\begin{pmatrix}-1&0&1\\0&0&0\\1&0&-1\end{pmatrix},$$
$$\mathbf{A}(\mathbf{A}+\mathbf{I})=\begin{pmatrix}-1&1&0\\1&-2&1\\0&1&-1\end{pmatrix}\begin{pmatrix}0&1&0\\1&-1&1\\0&1&0\end{pmatrix}=\begin{pmatrix}1&-2&1\\-2&4&-2\\1&-2&1\end{pmatrix},$$
$$\implies \mathrm{e}^{\mathbf{A}t}=\frac{1}{3}\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}-\mathrm{e}^{-t}\frac{1}{2}\begin{pmatrix}-1&0&1\\0&0&0\\1&0&-1\end{pmatrix}+\mathrm{e}^{-3t}\frac{1}{6}\begin{pmatrix}1&-2&1\\-2&4&-2\\1&-2&1\end{pmatrix}.$$

This is in accordance with the result of H71 b).

Secondly, we use the formula in b). From the solution of H69 we have $\lambda_1 = 0$, $\lambda_2 = -1$, $\lambda_3 = -3$ with corresponding normalized eigenvectors $\mathbf{v}_1 = \frac{1}{\sqrt{3}}(1,1,1)^\mathsf{T}$, $\mathbf{v}_2 = \frac{1}{\sqrt{2}}(1,0,-1)^\mathsf{T}$, $\mathbf{v}_3 = \frac{1}{\sqrt{6}}(1,-2,1)^\mathsf{T}$.

$$\Longrightarrow \mathrm{e}^{\mathbf{A}t} = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} + \mathrm{e}^{-t}\,\frac{1}{2}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix} + \mathrm{e}^{-3t}\,\frac{1}{6}\begin{pmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{pmatrix}.$$

For this note that the entries of $\mathbf{M} = \mathbf{v}\mathbf{v}^\mathsf{T} = (x_1, x_2, x_3)(x_1, x_2, x_3)^\mathsf{T}$ are $m_{ij} = x_i x_j$.