Name: ____________________ Student ID: ____________________ Group A

For each of the following problems, find the correct answer (tick as appropriate!). No justifications are required. Each problem has exactly one correct solution, which is worth 1 mark. Incorrect solutions (including no answer, multiple answers, or unreadable answers) will be assigned 0 marks; there are no penalties.

1. Which of the following ODE's has distinct solutions $y_1, y_2 \colon \mathbb{R} \to \mathbb{R}$ satisfying $y_1(0) = y_2(0)$ and $y_1'(0) = y_2'(0)$ ?
   ☐ $y'' = |y'|$ ☐ $y'' = \sqrt{t}\,y$ ■ $y'' = t\sqrt{y}$ ☐ $y'' = |y|$ ☐ $yy'' = 0$

2. The ODE $-2y\,\mathrm{d}x + x\,\mathrm{d}y = 0$ has the integrating factor
   ☐ $0$ ☐ $3/x$ ☐ $3/y$ ☐ $x^{-3/2}$ ■ $y^{-3/2}$

3. The solution of the IVP $y' = (y-1)(y-2)\cdots(y-2024),\ y(0) = \pi$ is
   ☐ increasing ■ decreasing ☐ convex ☐ concave
   ☐ none of the foregoing

4. For the solution $y(t)$ of the IVP $y' = \frac{2y+1}{t},\ y(1) = 2$ the value $y(2)$ is equal to
   ☐ $11/2$ ☐ $13/2$ ☐ $15/2$ ☐ $17/2$ ■ $19/2$

5. For the solution $y(t)$ of the IVP $y' = -t(y^2+1),\ y(0) = 1$ the value $y(1)$ is contained in
   ■ $[0, \frac{1}{2})$ ☐ $[\frac{1}{2}, 1)$ ☐ $[1, \frac{3}{2})$ ☐ $[\frac{3}{2}, 2)$ ☐ $[2, \infty)$

6. The power series $\sum_{n=1}^{\infty} n^n z^{n^2}$ has radius of convergence
   ☐ $0$ ☐ $1/\mathrm{e}$ ■ $1$ ☐ $\mathrm{e}$ ☐ $\infty$

7. The smallest integer $k$ such that $f_k(x) = \sum_{n=1}^{\infty} \frac{\cos(nx)}{n^k}$ is differentiable on $(0, 2\pi)$ is equal to
   ☐ $0$ ■ $1$ ☐ $2$ ☐ $3$ ☐ $4$

8. For which choice of $f_n(x)$ does the function sequence $(f_n)$ converge uniformly on $(0,1)$ ?
   ■ $\frac{\ln x}{n}$ ■ $\sqrt[n]{x}$ ■ $\frac{nx}{x+n}$ ■ $\mathrm{e}^{-nx^2}$ ■ $\frac{x}{n} \ln \frac{x}{n}$

9. The orthogonal trajectories of $y = C\mathrm{e}^x,\ C \in \mathbb{R}$ are
   ☐ $y = C\mathrm{e}^{-x}$ ☐ $2x - y^2 = C$ ☐ $x - y^2 = C$ ■ $2x + y^2 = C$
   ☐ $x + y^2 = C$
   with $C \in \mathbb{R}$.

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10. The sequence $\phi_0, \phi_1, \phi_2, \dots$ of Picard-Lindelöf iterates for the IVP $y' = \frac{1}{2}y^2,\ y(1) = 2$ has $\phi_2(t)$ equal to

■ $\frac{2}{3}t^3 + \frac{4}{3}$ ☐ $\frac{2}{3}t^3 - 2t^2 + 2t + \frac{4}{3}$ ☐ $\frac{2}{3}t^3 - 2$ ☐ $\frac{2}{3}t^3 - 2t^2 + 2t - \frac{2}{3}$
☐ $\frac{2}{3}t^3 + 2t^2 + 2t + 2$

11. $y'' + y' - 2y = \mathrm{e}^t + 1$ has a particular solution $y_p(t)$ of the form

☐ $(c_0 + c_1 t)\mathrm{e}^t$ ■ $c_0 + c_1 t\,\mathrm{e}^t$ ☐ $c_0 + c_1\mathrm{e}^t$ ☐ $c_0 t + c_1\mathrm{e}^t$ ☐ $c_0\mathrm{e}^t + c_1\mathrm{e}^{-2t}$
with constants $c_0, c_1 \in \mathbb{R}$.

12. The maximal solution of the IVP $y' = y^6 - 1,\ y(-2) = 0$ is defined on

☐ $(-\infty, -1)$ ☐ $(-1, 1)$ ☐ $[-1, 1]$ ☐ $(1, +\infty)$ ■ $(-\infty, +\infty)$

13. The matrix norm of $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ (subordinate to the Euclidean length on $\mathbb{R}^2$) is contained in the interval

☐ $[0, 1)$ ■ $[1, 2)$ ☐ $[2, 3)$ ☐ $[3, 4)$ ☐ $[4, \infty)$

14. The map $t \mapsto \begin{pmatrix} \frac{1}{2}(\mathrm{e}^{2t} + \mathrm{e}^{-2t}) & \mathrm{e}^{2t} - \mathrm{e}^{-2t} \\ \frac{1}{4}(\mathrm{e}^{2t} - \mathrm{e}^{-2t}) & \frac{1}{2}(\mathrm{e}^{2t} + \mathrm{e}^{-2t}) \end{pmatrix}$ is the matrix exponential function of

☐ $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ ☐ $\begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$ ☐ $\begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}$ ■ $\begin{pmatrix} 0 & 4 \\ 1 & 0 \end{pmatrix}$ ☐ $\begin{pmatrix} 0 & 1 \\ 4 & 0 \end{pmatrix}$

15. Which of the following defines a contraction of the interval $[1, 2]$ ?

☐ $x \mapsto (x^2 + 3)/4$ ☐ $x \mapsto 4 - 3x + x^2$ ☐ $x \mapsto (x^2 + 5)/12$
■ $x \mapsto (x^2 + 5)/6$ ☐ $x \mapsto 1/(2x)$

Time allowed: 60 min CLOSED BOOK ***Good luck!***

# Notes

Notes have only been written for Group A. In those places where Group B differs from Group A, the difference is indicated briefly at the end of the note.
The date of the midterm (2024/04/19) has been corrected for this version.
Question 8 inadvertently has 2 correct answers. (I had Answer E in mind but, as many students noted, Answer C is also correct.) Since this contradicts the stated rules of the game, every student receives 1 mark for Question 8.

**1** $y'' = t\sqrt{y}$ has, besides the all-zero function on $\mathbb{R}$, the solution $y(t) = \frac{1}{900}t^6$, as one easily finds using the power function Ansatz $y(t) = ct^r$. The ODE $yy'' = 0$ is equivalent to $y'' = 0$, which is solved by $y(t) = at + b$ ($a, b \in \mathbb{R}$) and has a unique solution for prescribed initial values $y(0)$, $y'(0)$. (This also follows from the Existence and Uniqueness Theorem, applied to $y'' = 0$.) The other three answers offered are explicit 2nd-order ODE's satisfying the assumptions of the Existence and Uniqueness Theorem, so that distinct solutions with the same initial values cannot exist. (In the case of $y'' = \sqrt{t}\,y$ solutions exist only on $[0, \infty)$.)

**2** Multiplying the ODE by $y^{-3/2}$ gives $-2y^{-1/2}\,\mathrm{d}x + xy^{-3/2}\,\mathrm{d}y = 0$ which of the form $P\,\mathrm{d}x + Q\,\mathrm{d}y$ with $P_y = y^{-3/2} = Q_x$ and hence exact on $\mathbb{R}^2 \setminus \{y = 0\}$. Answers B,C,D don't have this property. Answer A is also false: Zero is not considered as an integrating factor, since multiplication by zero renders the ODE useless.
In Group B the ODE was $y\,\mathrm{d}x - 2x\,\mathrm{d}y = 0$, which has the integrating factor $x^{-3/2}$.

**3** According to our discussion of the phase line, the (maximal) solution, which satisfies $y(0) \in (3,4)$, has domain $\mathbb{R}$ and range $(3,4)$. Since for $y \in (3,4)$ exactly 3 factors of $f(y) = (y-1)(y-2)\cdots(y-2024)$ are positive and 2021 negative, we have $y'(t) < 0$ for $t \in \mathbb{R}$, so that $y(t)$ is strictly decreasing. Answers C,D are wrong, because $y(t)$ has an inflection point: $y'' = f(y)' = f'(y)y'$, and between adjacent zeros of $f$ (in our case 3,4) there is always a zero $z$ of $f'$. If $t_0$ is such that $y(t_0) = z$, we have $y''(t_0) = 0$. Because $y(t)$ is decreasing, the curvature changes from concave to convex at $t_0$.
In Group B we have $y(0) \in (2,3)$, so that exactly 2 factors of $f(y) = (y-1)(y-2)\cdots(y-2024)$ are positive and $y'(t) > 0$ for $t \in \mathbb{R}$. Thus Answer A is correct for Group B (and Answers C,D likewise wrong).

**4** This ODE is 1st-order linear with associated homogeneous ODE $y' = \frac{2}{t}y$. The solution of the latter is

$$y_h(t) = c\exp\left(\int \frac{2}{t}\,\mathrm{d}t\right) = ct^2.$$

A particular solution of the inhomogeneous ODE is $y_p(t) = -1/2$ (shame on you if you haven't found it!), and hence the general solution is $y(t) = ct^2 - 1/2$, which has $y(1) = c - 1/2$. In Group A the initial condition $y(1) = 2$ gives $c = 5/2$, $y(2) = 19/2$, while in Group B $y(0) = 1$ gives $c = 3/2$, $y(2) = 11/2$.

**5** This is a separable ODE, which can be solved by the standard method (Group B comes first):

$$\begin{aligned}\frac{\mathrm{d}y}{y^2+1} &= -t\,\mathrm{d}t\\ \arctan y &= -t^2/2 + C\\ y &= \tan(C - t^2/2)\end{aligned}$$

$y(0) = \tan C = 1$ gives $C = \pi/4$, so that $y(t) = \tan(\pi/4 - t^2/2)$. It follows that $y(1) = \tan(\pi/4 - 1/2) \approx \tan(0.25) \approx 0.25 \in \left[0, \frac{1}{2}\right)$. (The exact value of $\tan(\pi/4 - 1/2)$ is $0.2934\ldots$.)

In Group A the computation is

$$\begin{aligned}
\left[\frac{1}{2}\ln\left(\eta^2-3\right)\right]_2^y &= [\ln\tau]_1^t \\
\frac{1}{2}\left(\ln\left(y^2-3\right)-\ln 1\right) &= \ln t \\
\ln\left(y^2-3\right) &= 2\ln t = \ln(t^2) \\
y^2-3 &= t^2 \\
y &= \sqrt{t^2+3},
\end{aligned}$$

and $y(2)=\sqrt{7}$.

**6** The standard form of this power series is $\sum_{k=1}^{\infty} a_k z^k$ with

$$a_k = \begin{cases} n^n & \text{if } k=n^2 \text{ is a perfect square,} \\ 0 & \text{if } k \text{ is not a perfect square.} \end{cases}$$

Since

$$\sqrt[k]{|a_k|} = \begin{cases} \sqrt[n^2]{n^n} = \sqrt[n]{n} & \text{if } k=n^2 \text{ is a perfect square,} \\ 0 & \text{if } k \text{ is not a perfect square,} \end{cases}$$

and $\sqrt[n]{n} \to 1$ for $n \to \infty$, the limit superior of the sequence $\left(\sqrt[k]{|a_k|}\right)$ is $L=1$.
$\Longrightarrow R = 1/L = 1$.

**7** In the lecture it was shown that $f_1(x) = \sum_{n=1}^{\infty} \frac{\cos(nx)}{n} = -\ln\left(2\sin\frac{x}{2}\right)$ for $x \in (0,2\pi)$. Clearly this function is differentiable. For $k \leq 0$ the series defining $f_k(x)$ doesn't converge anywhere.

**8** The correct answers are (C) and (E).
In (A) the limit function is $x \mapsto 0$, but $f_n(\mathrm{e}^{-n}) = -1$ for $n = 1,2,\dots$, showing that no uniform response to $\varepsilon = 1$ (and smaller values of $\varepsilon$) can exist.
In (B) the limit function is $x \mapsto 1$, but $f_n(1/2^n) = 1/2$ for $n = 1,2,\dots$, showing that no uniform response to $\varepsilon = 1/2$ can exist.
In (C) the limit function is $x \mapsto x$, and we have

$$\left|\frac{nx}{x+n} - x\right| = \left|\frac{-x^2}{x+n}\right| \leq \frac{1}{n} \quad \text{for } 0<x<1,$$

implying uniform convergence. (As uniform response to $\varepsilon > 0$ we can take $N = \lceil 1/\varepsilon/\rceil$.)
In (D) the limit function is $x \mapsto 0$, but $f_n(1/\sqrt{n}) = 1/\mathrm{e}$ for $n = 2,3,\dots$, showing that no uniform response to $\varepsilon = 1/\mathrm{e}$ can exist.
In (E) the limit function is $x \mapsto 0$, and $0 < x/n < 1/n$ and $\lim_{y\downarrow 0}(y\ln y) = 0$ imply uniform convergence. (If $\delta > 0$ is such that $|y\ln y| < \varepsilon$ for $0<y<\delta$, we can take $N = \lceil 1/\delta/\rceil$ as uniform response to $\varepsilon$.)

**9** Rewriting the equation as $y\mathrm{e}^{-x} = C$, we see that the curves are the contours of $f(x,y) = y\mathrm{e}^{-x}$ and hence satisfy the ODE

$$f_x\,\mathrm{d}x + f_y\,\mathrm{d}y = -\mathrm{e}^{-x}y\,\mathrm{d}x + \mathrm{e}^{-x}\,\mathrm{d}y = 0 \iff -y\,\mathrm{d}x + \mathrm{d}y = 0.$$

The orthogonal trajectories then satisfy $\mathrm{d}x + y\,\mathrm{d}y = 0$, which is exact (even separable) and solved by $x + y^2/2 = C$. Hence the correct answer is (D).

**10** $\phi_0(t) = 2$, $\phi_1(t) = 2 + \int_1^t \frac{1}{2}\phi_0(s)^2\,\mathrm{d}s = 2 + \int_1^t 2\,\mathrm{d}s = 2 + 2(t-1) = 2t$, $\phi_2(t) = 2 + \int_1^t \frac{1}{2}\phi_1(s)^2\,\mathrm{d}s = 2 + \int_0^t 2s^2\,\mathrm{d}s = 2 + \left[\frac{2}{3}s^3\right]_1^t = 2 + \frac{2}{3}t^3 - \frac{2}{3} = \frac{2}{3}t^3 + \frac{4}{3}$.

**11** This ODE has characteristic polynomial $X^2+X-2=(X-1)(X+2)$, which has roots $\lambda_1=1, \lambda_2=-2$, both with multiplicity $m=1$. Superposition gives a solution $y=y_1+y_2$ from solutions $y_1$ of $y''+y'-2y=1$ and $y_2$ of $y''+y'-2y=\mathrm{e}^t$. We can take $y_1(t)=-1/2$ and for $y_2$ use the Ansatz $y_2(t)=ct\,\mathrm{e}^t$, which after a short computation gives $c=1/3$. Hence (B) is the correct answer. (The general solution is $y(t)=-\frac{1}{2}+\frac{1}{3}t\,\mathrm{e}^t+c_1\mathrm{e}^t+c_2\mathrm{e}^{-2t}$, none of which fits any of the answers (A), (C), (D), (E).)

**12** The function $f(y)=y^6-1=(y^2-1)(y^4+y^2+1)$ has zeros $z_1=-1, z_2=1$. Since $y(-2)=0\in(z_1,z_2)$, the domain of the maximal solution is $(-\infty,+\infty)$ according to the theorem about the phase line from the lecture.

**13** $\|\mathbf{A}\|$ is equal to the square root of the largest eigenvalue of $\mathbf{A}^\mathsf{T}\mathbf{A}$, which in this case is $\left(\begin{smallmatrix}1&1\\1&2\end{smallmatrix}\right)$. This matrix has characteristic polynomial $\chi_\mathbf{A}(X)=X^2-3X+1$ and eigenvalues $\lambda_{1/2}=\frac{3\pm\sqrt{5}}{2}$.
$\implies \|\mathbf{A}\|=\sqrt{\frac{3+\sqrt{5}}{2}}=\frac{1+\sqrt{5}}{2}\approx 1.62$ (the golden ratio).
Alternatively we can reason as follows: For $\mathbf{x}=(0,1)^\mathsf{T}$ we have $|\mathbf{Ax}|/|\mathbf{x}|=\sqrt{2}$, implying $\|\mathbf{A}\|\geq\sqrt{2}$. On the other hand, $\|\mathbf{A}\|\leq\|\mathbf{A}\|_\mathrm{F}=\sqrt{3}$. Hence $\|\mathbf{A}\|\in\left[\sqrt{2},\sqrt{3}\right]$, and the correct answer must be (B).

**14** Calling the matrix function $\mathbf{\Phi}'(t)$, we have

$$\mathbf{\Phi}'(t)=\begin{pmatrix}\mathrm{e}^{2t}-\mathrm{e}^{-2t} & 2\mathrm{e}^{2t}+2\mathrm{e}^{-2t}\\ \frac{1}{2}(\mathrm{e}^{2t}+\mathrm{e}^{-2t}) & \mathrm{e}^{2t}-\mathrm{e}^{-2t}\end{pmatrix}=\begin{pmatrix}0&4\\1&0\end{pmatrix}\mathbf{\Phi}(t).$$

Since $t\mapsto\mathrm{e}^{\mathbf{A}t}$ solves the matrix ODE $\mathbf{Y}'=\mathbf{AY}$, the correct answer must be (D). (All but (D), (E) can also be excluded by the fact that for a diagonal matrix $\mathbf{A}$ the matrices $\mathrm{e}^{\mathbf{A}t}$ must also be diagonal.)

**15** This was probably the most difficult question. The correct answer is (D). The map $T\colon[1,2]\to\mathbb{R}$, $x\mapsto(x^2+5)/6$ is increasing with $T(1)=1$, $T(2)=3/2$, and hence maps $[1,2]$ into itself. The Mean Value Theorem gives

$$|T(x)-T(y)|=\left|T'(\xi)\right||x-y|=\frac{\xi}{3}|x-y| \quad \text{for } x,y\in[1,2],$$

where $\xi$ is some number between $x$ and $y$. Since $\xi\leq 2$, the map $T$ defines a contraction of $[1,2]$ with contraction constant $C=2/3$.
In (A), (B) the map $T$ satisfies $T'(2)=1$. The argument using the Mean Value Theorem gives $|T(x)-T(2)|=|T'(\xi)||x-2|$ with $\xi\in(x,2)$ for arbitrarily chosen $x\in[1,2]$. By choosing $x$ close to 2 we can make the factor $|T'(\xi)|$ arbitrarily close to 1. Thus $|T(x)-T(2)|\leq C|x-2|$ for a constant $C<1$ is impossible.
In (C), (E) the map $T$ satisfies $T(1)=1/2$ and hence doesn't map $[1,2]$ into itself.