Question 1 (ca. 12 marks)

Decide whether the following statements are true or false, and justify your answers.

a) There exists a solution $y(t)$ of $y' = \sin(y)^{29}$ satisfying $y(1) = 1$, $y(3) = 3$.

b) The maximal solution $y(t)$ of the initial value problem $y' = \sqrt{y^2 + t}$, $y(0) = 1$ is defined at time $t = 2023$.

c) The IVP $(x^4 - 1)y'' + (x^2 - 1)y' + (x - 1)y = 0$, $y(0) = y'(0) = 1$ has a solution $y(x)$ that is defined at $x = 4$.

d) If $\mathbf{A} \in \mathbb{R}^{n\times n}$ satisfies $\mathbf{A}^3 = \mathbf{0}$ but $\mathbf{A}^2 \neq \mathbf{0}$ then at least one entry (coordinate function) of $e^{\mathbf{A}t}$ is a polynomial in t of exact degree 2.

e) The function $\mathbb{R} \to \mathbb{R}$, $t \mapsto t^2 \cos t$ satisfies an ODE of the form $y''' + a_2y'' + a_1y' + a_0y = 0$ with $a_0, a_1, a_2 \in \mathbb{R}$.

f) Every point of $\mathbb{R}^2$ is on a unique integral curve of $(y-x^2)\,\mathrm{d}x + (x-y^2)\,\mathrm{d}y = 0$.

Question 2 (ca. 10 marks)

Consider the differential equation

$$x^2y'' + 2\,x^2y' - 2y = 0. \qquad \text{(DE)}$$

a) Verify that $x_0 = 0$ is a regular singular point of (DE).

b) Determine the general solution of (DE) on $(0, \infty)$.

c) Using the result of b), state the general solution of (DE) on $(-\infty, 0)$ and on $\mathbb{R}$.

Question 3 (ca. 5 marks)

For the initial value problem

$$y' = \frac{y^2 + 2}{2y + ty}, \qquad y(0) = 2$$

determine the maximal solution $y(t)$ and its domain.

Question 4 (ca. 10 marks)

Consider $\mathbf{A} = \begin{pmatrix} -6 & -8 & -16 \\ -5 & -8 & -14 \\ 5 & 7 & 13 \end{pmatrix}$ and $\mathbf{b}(t) = \begin{pmatrix} 4t \\ 2 - 2t \\ 1 + t \end{pmatrix}$.

a) Determine a real fundamental system of solutions of the system $\mathbf{y}' = \mathbf{A}\mathbf{y}$.

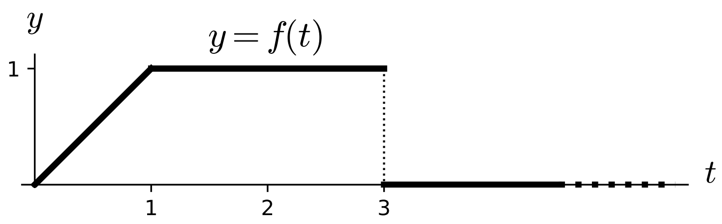
b) Solve the initial value problem $\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{b}(t)$, $\mathbf{y}(0) = (0, 0, 0)^{\mathsf{T}}$.
Hint: There is a particular solution of the form $\mathbf{y}(t) = \mathbf{w}_0 + t\,\mathbf{w}_1$ ($\mathbf{w}_0, \mathbf{w}_1 \in \mathbb{R}^3$).

Question 5 (ca. 6 marks)

For the function f sketched below, solve the initial value problem

$$y'' + 4\,y = f(t), \quad y(0) = 0,\ y'(0) = 1$$

with the Laplace transform.



Notes: For the solution $y(t)$ explicit formulas valid in the intervals $[0,1]$, $[1,3]$, $[3,\infty)$ are required. You *must* use the Laplace transform for the computation.

Question 6 (ca. 7 marks)

a) Determine a real fundamental system of solutions of

$$y^{(4)} + y'' - 36\,y' + 52\,y = 0.$$

b) Determine the general real solution of

$$y^{(4)} + y'' - 36\,y' + 52\,y = 4 + 5\,\mathrm{e}^{2t} - \sin t.$$

Solutions

1 a) False. From $y(3) = y(1) + \int_1^3 y'(t)\,\mathrm{d}t = y(1) + \int_1^3 \sin^{29} y(t)\,\mathrm{d}t$ we obtain $\int_1^3 \sin^{29} y(t)\,\mathrm{d}t = 2$. Since the integrand is continuous and ≤ 1, this can only hold if $\sin^{29} y(t) = 1$ for $t \in [1,3]$, i.e., $\sin y(t) = 1$ for $t \in [1,3]$, which is already false for $t = 1$. [2]

b) True. Denoting the maximal domain by (a,b), we have $y'(t) > 0$ for $t \in [0,b)$, i.e., $y(t)$ is increasing on $[0,b)$. Thus $y(t)$ is bounded from below by $y_1(t) \equiv 1$, which exists for all $t \geq 0$. On the other hand, we have $y' \leq \sqrt{y^2 + 2yt + t^2} = y + t$ (since $y \geq 1$) for those t, and hence $y(t)$ is bounded from above by the solution $y_2(t)$ of the linear ODE $y' = y + t$, which also exists for all $t \geq 0$. This shows that $y(t)$ cannot escape to $\pm\infty$ in finite time, and hence must exist for all $t \geq 0$. [2]

c) True. Every solution of $(x^3 + x^2 + x + 1)\,y'' + (x+1)y' + y = 0$, $y(0) = y'(0) = 1$ solves the original IVP (just multiply by $x - 1$), and the maximal solution of this new IVP has domain at least $(-1, \infty)$, as follows from $x^3 + x^2 + x + 1 = (x+1)(x^2+1)$ and the sharpened version of the EUT in the linear case. [2]

d) True. Plugging $\mathbf{A}$ into the exponential series or using the new method for computing $\mathrm{e}^{\mathbf{A}t}$ gives $\mathrm{e}^{\mathbf{A}t} = \mathbf{I} + t\mathbf{A} + (t^2/2)\mathbf{A}^2$ (when using the new method, check that $1, t, t^2/2$ solve $\mu_{\mathbf{A}}(\mathrm{D})y = \mathrm{D}^3 y = 0$ and satisfy the required initial conditions). Thus all entries of $\mathrm{e}^{\mathbf{A}t}$ are polynomials of degree ≤ 2. By assumption, $\mathbf{A}^2$ has a nonzero entry in some position (i,j), and hence the (i,j) entry of $\mathrm{e}^{\mathbf{A}t}$ has degree exactly 2. [2]

e) False. Writing $a(X) = X^3 + a_2 X^2 + a_1 X + a_0$, suppose by contradiction that $a(\mathrm{D})\,[t^2 \cos t] = 0$. Since $t^2 \cos t = \operatorname{Re}\left(t^2 \mathrm{e}^{\mathrm{i}t}\right)$ and $t \mapsto t^2 \mathrm{e}^{\mathrm{i}t}$ satisfies $\left(\mathrm{D}^2 + 1\right)^3 y = 0$, which has real coefficients, we have $\left(\mathrm{D}^2+1\right)^3 [t^2 \cos t] = 0$ as well. This implies $d(\mathrm{D})y = 0$ for $d(X) = \gcd\left((X^2+1)^3, a(X)\right) \in \{1, X^2+1\}$. But $t \mapsto t^2 \cos t$ is not the all-zero function and doesn't satisfy $y'' + y = 0$ (as is easily verified); contradiction. [2]

f) False. The ODE has two singular points, viz. $(0,0)$ and $(1,1)$, and hence points of $\mathbb{R}^2$ are not guaranteed to be on exactly one integral curve. [1]

In fact the differential 1-form $\omega = (y - x^2)\,\mathrm{d}x + (x - y^2)\,\mathrm{d}y$ is exact, $\omega = \mathrm{d}f$ for $f(x,y) = xy - \frac{1}{3}x^3 - \frac{1}{3}y^3$, so that the solution in implicit form is $f(x,y) = C$. It can be checked that f has a strict local maximum in $(1,1)$, and hence $(1,1)$ is on no integral curve at all. [1]

Remarks: No marks were assigned for answers without justification.

$$\sum_1 = 12$$

2 a) The explicit form of (DE) is

$$y'' + 2\,y' - \frac{2}{x^2}\,y = 0$$

$p(x) := 2$ is analytic at 0, and $q(x) := -\frac{2}{x^2}$ has a pole of order 2 at 0. This shows that 0 is a regular singular point of (DE). [1]

Alternatively, use that the limits defining p_0, q_0 below are finite.

b) From a) we have $p_0 = \lim_{x\to 0} x\,p(x) = 0$, $q_0 = \lim_{x\to 0} x^2\,q(x) = -2$. (These coefficients can just be read off from the explicit form.)
$\Longrightarrow$ The indicial equation is

$$r^2 + (p_0 - 1)r + q_0 = r^2 - r - 2 = (r+1)(r-2) = 0.$$

$\Longrightarrow$ The exponents at the singularity $x_0 = 0$ are $r_1 = 2$, $r_2 = -1$. $\boxed{1}$

Since $r_1 - r_2 \in \mathbb{Z}$, only one fractional series solution corresponding to the larger exponent $r_1 = 2$ is guaranteed. However, it turns out that nevertheless there exist two fundamental solutions y_1, y_2 of the form

$$\begin{aligned} y_1(x) &= x^2 \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+2}, \\ y_2(x) &= x^{-1} \sum_{n=0}^{\infty} b_n x^n = \sum_{n=0}^{\infty} b_n x^{n-1} \end{aligned}$$

with normalization $a_0 = b_0 = 1$.

First we determine $y_1(x)$. We have

$$\begin{aligned} 0 &= x^2 y_1'' + 2\,x^2 y_1' - 2\,y_1 \\ &= x^2 \sum_{n=0}^{\infty} (n+2)(n+1) a_n x^n + 2\,x^2 \sum_{n=0}^{\infty} (n+2) a_n x^{n+1} - 2 \sum_{n=0}^{\infty} a_n x^{n+2} \\ &= x^2 \sum_{n=0}^{\infty} (n+2)(n+1) a_n x^n + 2\,x^2 \sum_{n=1}^{\infty} (n+1) a_{n-1} x^n - 2 \sum_{n=0}^{\infty} a_n x^{n+2} \\ &= (2-2) a_0 x^2 + \sum_{n=1}^{\infty} \big[\big((n+2)(n+1) - 2\big) a_n + 2(n+1) a_{n-1}\big]\, x^{n+2} \\ &= \sum_{n=1}^{\infty} \big[(n^2 + 3n) a_n + 2(n+1) a_{n-1}\big]\, x^{n+2}. \end{aligned}$$

Equating coefficients gives the recurrence relation

$$a_n = -\frac{2(n+1)}{n(n+3)} a_{n-1} \quad \text{for } n = 1, 2, 3, \dots, \qquad \boxed{1}$$

and with $a_0 = 1$ further $a_n = (-1)^n \frac{2^n (n+1)}{4\cdot 5\cdot 6\cdots (n+3)}$ for $n \geq 1$. Replacing a_n by $a_n/6$, which doesn't affect the property of being a (fundamental) solution, gives the slightly simplified solution

$$\begin{aligned} y_1(x) &= \sum_{n=0}^{\infty} (-1)^n \frac{2^n (n+1)}{(n+3)!}\, x^{n+2} \qquad \boxed{1\tfrac{1}{2}} \\ &= \frac{1}{6} \left(x^2 - x^3 + \frac{2^2 \cdot 3}{4 \cdot 5}\, x^4 - \frac{2^3 \cdot 4}{4 \cdot 5 \cdot 6}\, x^5 + \frac{2^4 \cdot 5}{4 \cdot 5 \cdot 6 \cdot 7}\, x^6 \mp \cdots \right) \end{aligned}$$

For the determination of $y_2(x)$ we repeat the process with exponents decrreased by 3:

$$\begin{aligned}
0 &= x^2\,y_2'' + 2\,x^2 y_2' - 2\,y_2\\
&= x^2\sum_{n=0}^{\infty}(n-1)(n-2)b_n x^{n-3} + 2\,x^2\sum_{n=0}^{\infty}(n-1)b_n x^{n-2} - 2\sum_{n=0}^{\infty} b_n x^{n-1}\\
&= x^2\sum_{n=0}^{\infty}(n-1)(n-2)b_n x^{n-3} + 2\,x^2\sum_{n=1}^{\infty}(n-2)b_{n-1} x^{n-3} - 2\sum_{n=0}^{\infty} b_n x^{n-1}\\
&= (2-2)a_0x^2 + \sum_{n=1}^{\infty}\left[\big((n-1)(n-2)-2\big)b_n + 2(n-2)b_{n-1}\right]x^{n-1}\\
&= \sum_{n=1}^{\infty}\left[(n^2-3n)b_n + 2(n-2)b_{n-1}\right]x^{n-1}.
\end{aligned}$$

Here we obtain the recurrence relation $b_1 = -b_0$, $b_2 = 0$, b_3 arbitrary, and

$$b_n = -\frac{2(n-2)}{n(n-3)}\,b_{n-1} \quad \text{for } n = 4,5,6,\dots. \qquad \boxed{1}$$

Setting $b_0 = 1$, $b_3 = 0$ gives $b_n = 0$ for all $n \geq 2$ and hence the rational function solution

$$y_2(x) = x^{-1}(1-x) = \frac{1}{x} - 1. \qquad \boxed{1\tfrac{1}{2}}$$

Alternative solution: We use the general recurrence relation for the rational functions $a_n(r)$, viz. $a_0(r) = 1$ and

$$a_n(r) = -\frac{1}{F(r+n)}\sum_{k=0}^{n-1}\left[(r+k)p_{n-k} + q_{n-k}\right]a_k(r) \quad \text{for } n \geq 1. \qquad \text{(RR)}$$

Since $F(r) = (r+1)(r-2)$ and all coefficients p_i, q_i except for $p_1 = 2$, $q_0 = -2$ are zero, we obtain

$$\begin{aligned}
a_n(r) &= -\frac{(r+n-1)2}{(r+n+1)(r+n-2)}\,a_{n-1}(r)\\
&= \dots = \frac{(-1)^n 2^n (r+n-1)}{(r-1)(r+2)(r+3)\cdots(r+n+1)} \quad \text{for } n \geq 1,\ r \neq 1,-2,-3,\dots
\end{aligned}$$

Thus the coefficients $y_1(x)$ are $a_n(2) = \frac{(-1)^n 2^n (n+1)}{4\cdot 5\cdots(n+3)}$ (the same as in the unsimplified solution above), and those of $y_2(x)$ are $a_n(-1) = \frac{(-1)^{n-1}2^{n-1}(n-2)}{n!}$ (valid also for $n = 0$).

$$\begin{aligned}
\implies y_2(x) &= \sum_{n=0}^{\infty} a_n(-1)x^{n-1} = x^{-1} - 1 + 0\,x + \sum_{n=3}^{\infty}\frac{(-1)^{n-1}2^{n-1}(n-2)}{n!}\,x^{n-1}\\
&= x^{-1} - 1 + \sum_{n=0}^{\infty}\frac{(-1)^{n+2}2^{n+2}(n+1)}{(n+3)!}\,x^{n+2}\\
&= x^{-1} - 1 + 4\sum_{n=0}^{\infty}\frac{(-1)^n 2^n (n+1)}{(n+3)!}\,x^{n+2} = x^{-1} - 1 + 4\,y_1(x),
\end{aligned}$$

where $y_1(x)$ is the simplified solution above. Since $x \mapsto x^{-1} - 1$ is a solution, the newly determined functions $y_1(x)$ and $y_2(x)$ form just another basis of the solution space (as is also postulated by the general theory).

The general (real) solution on $(0, \infty)$ is then $y(x) = c_1\, y_1(x) + c_2\, y_2(x)$, $c_1, c_2 \in \mathbb{R}$. $\boxed{\frac{1}{2}}$

That solutions are defined on the whole of $(0, \infty)$, is guaranteed by the analyticity of $p(x)$, $q(x)$ in $\mathbb{C} \setminus \{0\}$, but follows also readily from the easily established fact that the radius of convergence of the power series fundamental solution (or both power series in the alternative solution) is ∞. $\boxed{\frac{1}{2}}$

c) The solution on $(-\infty, 0)$ is $y(x) = c_1\, y_1(x) + c_2\, y_2^-(x)$ with the same power series $y_1(x)$ as in b) and

$$y_2^-(x) = |x|^{-1}(1 - x) = -x^{-1}(1 - x) = 1 - \frac{1}{x}. \qquad \boxed{1}$$

Since changing the sign of $y_2^-(x)$ doesn't change the span $\langle y_1, y_2^- \rangle$, we can just use the same functions as in b), with domain extended to $\mathbb{R} \setminus \{0\}$.

Since $y_1(x)$ is analytic at zero but $y_2(x)$ is not, the general solution on $\mathbb{R}$ is $y(x) = c\, y_1(x)$, $c \in \mathbb{R}$. $\boxed{1}$

Remarks: In b) several students falsely obtained $p_0 = 2$ (p_0 is the coefficient of x^{-1} in $p(x)$!), which changes the indicial equation to $r^2 + r - 2 = (r-1)(r+2) = 0$. Afterwards there is no continuation, because with the wrong exponents r_1, r_2 the recurrence relation can't be solved.

A few students obtained in b) correctly $y_2(x) = 1/x - 1$, but then in c) used $y_2(-x) = -1/x - 1$ (or $1/x + 1$) on $(-\infty, 0)$, which is wrong.

The "alternative solution" in b) is actually the shorter and less error prone one, if you think for a while about it. It is equivalent to deriving the recurrence relation for the coefficients of $y(x) = x^r \times$ "power series" for general r by equating coefficients and then substituting $r = r_1$ and $r = r_2$ into it. The necessary computation is only slightly more costly than that done for one of $y_1(x)$, $y_2(x)$ above. Moreover, the general recurrence relation (RR) has already been derived in the lecture and can just be quoted if you have it on your cheat sheet.

$$\sum_2 = 10$$

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3 Since $\frac{y^2+2}{2y+ty} = \frac{y^2+2}{y}\frac{1}{2+t}$, this is a separable equation and can be solved by the usual method:

$$\begin{aligned}
\frac{y\,\mathrm{d}y}{y^2+2} &= \frac{\mathrm{d}t}{t+2} && \boxed{1}\\
\frac{1}{2}\ln\left(y^2+2\right) &= \ln|t+2| + C\\
\ln\left(y^2+2\right) &= 2\ln|t+2| + C' = \ln\left((t+2)^2\right) + C'\\
y^2 + 2 &= \mathrm{e}^{C'}(t+2)^2 = C''(t+2)^2 && (C'' > 0)\\
y &= \pm\sqrt{C''(t+2)^2 - 2} && \boxed{2}
\end{aligned}$$

The initial value $y(0) = 2$ gives $\pm\sqrt{4C'' - 2} = 2$, i.e., $C'' = \frac{3}{2}$ and

$$y(t) = +\sqrt{\tfrac{3}{2}(t+2)^2 - 2} = \sqrt{\tfrac{3}{2}\,t^2 + 6t + 5}. \qquad \boxed{1}$$

The right-hand side is defined and differentiable for $(t+2)^2 > \frac{4}{3}$, i.e., $t < -2 - \frac{2}{\sqrt{3}}$ or $t > -2 + \frac{2}{\sqrt{3}}$. Since 0 is contained in $\left(-2 + \frac{2}{\sqrt{3}}, \infty\right)$, the maximal domain of $y(t)$ is $\left(-2 + \frac{2}{\sqrt{3}}, \infty\right)$. $\boxed{1}$.

Remarks: Many students didn't recognize that this ODE is separable and used one of the other „Ansaetze" we have discussed for finding a solution. Since the ODE isn't homogeneous, the substitution $z(t) = y(t)/t$ won't work (at least not simplify anything), and neither do substitutions of the form $z(t) = y(t)^r$.

You were required to determine the solution in explicit form (1 mark penalty for implicit solutions), and state the exact domain of the maximal solution, which must be an open interval (1 mark penalty for two disjoint intervals, 0.5 marks penalty for including the end point).

$$\sum_3 = 5$$

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4 a) The characteristic polynomial of $\mathbf{A}$ is

$$\chi_{\mathbf{A}}(X) = (-1)^3 \begin{vmatrix} -6-X & -8 & -16 \\ -5 & -8-X & -14 \\ 5 & 7 & 13-X \end{vmatrix} = -\begin{vmatrix} -6-X & -8 & -16 \\ 0 & -1-X & -1-X \\ 5 & 7 & 13-X \end{vmatrix}$$

$$= (X+1)\begin{vmatrix} -6-X & -8 & -16 \\ 0 & 1 & 1 \\ 5 & 7 & 13-X \end{vmatrix} = (X+1)\begin{vmatrix} -6-X & -8 & -8 \\ 0 & 1 & 0 \\ 5 & 7 & 6-X \end{vmatrix}$$

$$= (X+1)\left[(-6-X)(6-X) - 5(-8)\right] = (X+1)(X^2+4)$$

$$= (X+1)(X-2\mathrm{i})(X+2\mathrm{i}).$$

$\Longrightarrow$ The eigenvalues of $\mathbf{A}$ are $\lambda_1 = -1$, $\lambda_2 = 2\mathrm{i}$, $\lambda_3 = -2\mathrm{i}$, all with algebraic multiplicity 1. $\boxed{2}$

$$\mathbf{A} + \mathbf{I} = \begin{pmatrix} -5 & -8 & -16 \\ -5 & -7 & -14 \\ 5 & 7 & 14 \end{pmatrix} \to \begin{pmatrix} -5 & -8 & -16 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

$\Longrightarrow$ The eigenspace corresponding to $\lambda_1 = -1$ is generated by $\mathbf{v}_1 = (0, -2, 1)^{\mathsf{T}}$.

$$\mathbf{A} - 2\mathrm{i}\,\mathbf{I} = \begin{pmatrix} -6-2\mathrm{i} & -8 & -16 \\ -5 & -8-2\mathrm{i} & -14 \\ 5 & 7 & 13-2\mathrm{i} \end{pmatrix} \to \begin{pmatrix} 5 & 7 & 13-2\mathrm{i} \\ 0 & -1-2\mathrm{i} & -1-2\mathrm{i} \\ 0 & \frac{2+14\mathrm{i}}{5} & \frac{2+14\mathrm{i}}{5} \end{pmatrix}$$

$$\to \begin{pmatrix} 5 & 7 & 13-2\mathrm{i} \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 5 & 0 & 6-2\mathrm{i} \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

During the elimination we have used that $\frac{6+2i}{5}7-8=\frac{42+14i-40}{5}=\frac{2+14i}{5}$ and $\frac{6+2i}{5}(13-2i)-16=\frac{82+14i-80}{5}=\frac{2+14i}{5}$. This computation can in fact be omitted, because the rank of $\mathbf{A}-2i\,\mathbf{I}$ must be 2, so that the last row of the second matrix must be a linear combination of the first two rows.

$\Longrightarrow$ The eigenspace corresponding to $\lambda_2=2i$ is generated by $\mathbf{v}_2=\left(\frac{6-2i}{5},1,-1\right)^{\mathsf{T}}$.

Since $\mathbf{A}$ is real, the eigenspace corresponding to $\lambda_3=-2i$ is then generated by $\mathbf{v}_3=\overline{\mathbf{v}}_2=\left(\frac{6+2i}{5},1,-1\right)^{\mathsf{T}}$.

A real fundamental system of solutions is then

$$\begin{aligned}
\mathbf{y}_1(t)&=e^{-t}\begin{pmatrix}0\\-2\\1\end{pmatrix},\\
\mathbf{y}_2(t)&=\operatorname{Re}\left(e^{2it}\begin{pmatrix}(6-2i)/5\\1\\-1\end{pmatrix}\right)=\begin{pmatrix}\frac{6}{5}\cos(2t)+\frac{2}{5}\sin(2t)\\\cos(2t)\\-\cos(2t)\end{pmatrix},\\
\mathbf{y}_3(t)&=\operatorname{Im}\left(e^{2it}\begin{pmatrix}(6-2i)/5\\1\\-1\end{pmatrix}\right)=\begin{pmatrix}\frac{6}{5}\sin(2t)-\frac{2}{5}\cos(2t)\\\sin(2t)\\-\sin(2t)\end{pmatrix}
\end{aligned}$$ 3

b) $\mathbf{y}(t)=\mathbf{w}_0+t\,\mathbf{w}_1$ is a solution iff

$$\begin{aligned}
\mathbf{w}_1=\mathbf{y}'(t)=\mathbf{A}\mathbf{y}(t)+\mathbf{b}(t)&=\mathbf{A}\left(\mathbf{w}_0+t\,\mathbf{w}_1\right)+\begin{pmatrix}4t\\2-2t\\1+t\end{pmatrix}\\
&=\mathbf{A}\mathbf{w}_0+\begin{pmatrix}0\\2\\1\end{pmatrix}+t\left(\mathbf{A}\mathbf{w}_1+\begin{pmatrix}4\\-2\\1\end{pmatrix}\right)\\
\iff\quad \mathbf{w}_1&=\mathbf{A}\mathbf{w}_0+\begin{pmatrix}0\\2\\1\end{pmatrix}\ \wedge\ \mathbf{A}\mathbf{w}_1+\begin{pmatrix}4\\-2\\1\end{pmatrix}=\mathbf{0}\\
\iff\quad \mathbf{A}\mathbf{w}_1&=\begin{pmatrix}-4\\2\\-1\end{pmatrix}\ \wedge\ \mathbf{A}\mathbf{w}_0=\mathbf{w}_1-\begin{pmatrix}0\\2\\1\end{pmatrix}
\end{aligned}$$ 1

Since $\mathbf{A}$ is invertible, we can solve these systems, in order, to obtain first $\mathbf{w}_1$ and then $\mathbf{w}_0$.

$$\left(\begin{array}{ccc|c}-6&-8&-16&-4\\-5&-8&-14&2\\5&7&13&-1\end{array}\right)\to\left(\begin{array}{ccc|c}-1&-1&-3&-5\\0&-1&-1&1\\5&7&13&-1\end{array}\right)\to\left(\begin{array}{ccc|c}-1&-1&-3&-5\\0&-1&-1&1\\0&2&-2&-26\end{array}\right)$$
$$\to\left(\begin{array}{ccc|c}1&1&3&5\\0&-1&-1&1\\0&0&-4&-24\end{array}\right)\to\left(\begin{array}{ccc|c}1&1&3&5\\0&1&1&-1\\0&0&1&6\end{array}\right)$$

$\Longrightarrow x_3 = 6$, $x_2 = -1 - x_3 = -7$, $x_1 = 5 - x_2 - 3x_3 = -6$, so that $\mathbf{w}_1 = (-6, -7, 6)^\mathsf{T}$.

$$\left(\begin{array}{rrr|r} -6 & -8 & -16 & -6 \\ -5 & -8 & -14 & -9 \\ 5 & 7 & 13 & 5 \end{array}\right) \to \left(\begin{array}{rrr|r} -1 & -1 & -3 & -1 \\ 0 & -1 & -1 & -4 \\ 5 & 7 & 13 & 5 \end{array}\right) \to \left(\begin{array}{rrr|r} -1 & -1 & -3 & -1 \\ 0 & -1 & -1 & -4 \\ 0 & 2 & -2 & 0 \end{array}\right)$$

$$\to \left(\begin{array}{rrr|r} 1 & 1 & 3 & 1 \\ 0 & -1 & -1 & -4 \\ 0 & 0 & -4 & -8 \end{array}\right) \to \left(\begin{array}{rrr|r} 1 & 1 & 3 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

$\Longrightarrow x_3 = 2$, $x_2 = 4 - x_3 = 2$, $x_1 = 1 - x_2 - 3x_3 = -7$, so that $\mathbf{w}_0 = (-7, 2, 2)^\mathsf{T}$.

$$\Longrightarrow \mathbf{y}_p(t) = \begin{pmatrix} -7 - 6t \\ 2 - 7t \\ 2 + 6t \end{pmatrix} \quad \text{is a particular solution} \qquad \boxed{2}$$

The general solution of $\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{b}(t)$ is $\mathbf{y}(t) = \mathbf{y}_p(t) + c_1\mathbf{y}_1(t) + c_2\mathbf{y}_2(t) + c_3\mathbf{y}_3(t)$. In order to satisfy the required initial condition, (c_1, c_2, c_3) needs to solve

$$\left(\begin{array}{rrr|r} 0 & 6/5 & -2/5 & 7 \\ -2 & 1 & 0 & -2 \\ 1 & -1 & 0 & -2 \end{array}\right) \to \left(\begin{array}{rrr|r} 0 & 6 & -2 & 35 \\ -2 & 1 & 0 & -2 \\ -1 & 0 & 0 & -4 \end{array}\right) \to \left(\begin{array}{rrr|r} 0 & -6 & 2 & -35 \\ -2 & 1 & 0 & -2 \\ 1 & 0 & 0 & 4 \end{array}\right)$$

$\Longrightarrow c_1 = 4$, $c_2 = -2 + 2\,c_1 = 6$, $c_3 = (-35 + 6\,c_2)/2 = 1/2$, $\boxed{2}$
and the final answer is

$$y(t) = \begin{pmatrix} -7 - 6t \\ 2 - 7t \\ 2 + 6t \end{pmatrix} + 4\,\mathrm{e}^{-t} \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} + 6 \begin{pmatrix} \frac{6}{5}\cos(2t) + \frac{2}{5}\sin(2t) \\ \cos(2t) \\ -\cos(2t) \end{pmatrix} + \frac{1}{2} \begin{pmatrix} \frac{6}{5}\sin(2t) - \frac{2}{5}\cos(2t) \\ \sin(2t) \\ -\sin(2t) \end{pmatrix}$$

$$= \begin{pmatrix} -7 - 6t + 7\cos(2t) + 3\sin(2t) \\ 2 - 7t - 8\,\mathrm{e}^{-t} + 6\cos(2t) + \frac{1}{2}\sin(2t) \\ 2 + 6t + 4\,\mathrm{e}^{-t} - 6\cos(2t) - \frac{1}{2}\sin(2t) \end{pmatrix}. \qquad \boxed{1+}$$

Remarks: In a) several students overlooked the requirement "real fundamental system" and used a complex fundamental system instead (1 mark penalty).

b) wasn't solved by many students, the most frequent error being a confusion of the present conditions for $\mathbf{w}_0, \mathbf{w}_1$ with those in the first sample exam. (In the sample exam the system was of the form $\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{b}$ with a constant vector $\mathbf{b}$, but here it is not!)

$$\sum_4 = 11$$

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5 Writing $Y(s) = \mathcal{L}\{y(t)\}$, $F(s) = \mathcal{L}\{f(t)\}$, and applying the Laplace transform to both sides of the ODE gives

$$\begin{aligned} \mathcal{L}\{y'' + 4\,y\} &= s^2\,Y(s) - s\,y(0) - y'(0) + 4\,Y(s) \\ &= (s^2 + 4)Y(s) - 1 = \mathcal{L}\big\{f(t)\big\} = F(s). \end{aligned}$$

Further we have

$$\begin{aligned} f(t) &= t\big(\mathrm{u}(t) - \mathrm{u}(t-1)\big) + \mathrm{u}(t-1) - \mathrm{u}(t-3) \\ &= t\mathrm{u}(t) - (t-1)\mathrm{u}(t-1) - \mathrm{u}(t-3), & \boxed{1} \\ \implies F(s) &= \frac{1}{s^2} - \frac{\mathrm{e}^{-s}}{s^2} - \frac{\mathrm{e}^{-3s}}{s}. & \boxed{1} \\ \implies Y(s) &= \frac{1+F(s)}{s^2+4} = \frac{1}{s^2+4} - \frac{\mathrm{e}^{-3s}}{s(s^2+4)} + \frac{1-\mathrm{e}^{-s}}{s^2(s^2+4)} & \boxed{1} \\ &= \frac{1}{s^2+4} - \mathrm{e}^{-3s}\left(\frac{1}{4s} - \frac{s}{4(s^2+4)}\right) + \left(1-\mathrm{e}^{-s}\right)\left(\frac{1}{4s^2} - \frac{1}{4(s^2+4)}\right) \\ &= \frac{1}{4s^2} + \frac{3}{4(s^2+4)} - \mathrm{e}^{-3s}\left(\frac{1}{4s} - \frac{s}{4(s^2+4)}\right) - \mathrm{e}^{-s}\left(\frac{1}{4s^2} - \frac{1}{4(s^2+4)}\right) \end{aligned}$$

Since $\mathcal{L}\{\cos(2t)\} = \frac{s}{s^2+4}$, $\mathcal{L}\{\sin(2t)\} = \frac{2}{s^2+4}$ (cf. lecture), we have $\mathcal{L}^{-1}\left\{\frac{1}{4s} - \frac{s}{4(s^2+4)}\right\} = \frac{1}{4} - \frac{1}{4}\cos(2t)$, $\mathcal{L}^{-1}\left\{\frac{1}{4s^2} - \frac{1}{4(s^2+4)}\right\} = \frac{1}{4}t - \frac{1}{8}\sin(2t)$, and hence

$$\implies y(t) = \frac{t}{4} + \frac{3}{8}\sin(2t) - \mathrm{u}(t-3)\left(\frac{1}{4} - \frac{1}{4}\cos\big(2(t-3)\big)\right) - \mathrm{u}(t-1)\left(\frac{t-1}{4} - \frac{1}{8}\sin\big(2(t-1)\big)\right) \quad \boxed{3}$$

$$= \begin{cases} \frac{1}{4}t + \frac{3}{8}\sin(2t) & \text{for } 0 \leq t \leq 1, \\ \frac{1}{4} + \frac{3}{8}\sin(2t) + \frac{1}{8}\sin(2t-2) & \text{for } 1 \leq t \leq 3, \\ \frac{3}{8}\sin(2t) + \frac{1}{8}\sin(2t-2) + \frac{1}{4}\cos(2t-6) & \text{for } t \geq 3. \end{cases} \quad \boxed{1}$$

Remarks: Quite a few students had problems with this question, although we have practised this type of question a lot (both in the regular exercises and the sample exams).

$$\sum_5 = 7$$

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6 a) The characteristic polynomial is

$$\begin{aligned} a(X) &= X^4 + X^2 - 36\,X + 52 \\ &= (X-2)(X^3 + 2\,X^2 + 5\,X - 26) \\ &= (X-2)^2(X^2 + 4\,X + 13) \\ &= (X-2)^2(X+2-3\mathrm{i})(X+2+3\mathrm{i}) & \boxed{1} \end{aligned}$$

with zeros $\lambda_1 = 2$ of multiplicity 2 and $\lambda_{2/3} = -2 \pm 3\mathrm{i}$ of multiplicity 1.

$\Longrightarrow$ A complex fundamental system of solutions is e^{2t}, $t\,\mathrm{e}^{2t}$, $\mathrm{e}^{(-2+3\mathrm{i})t}$, $\mathrm{e}^{(-2-3\mathrm{i})t}$, and the corresponding real fundamental system is

$$\mathrm{e}^{2t}, \quad t\,\mathrm{e}^{2t}, \quad \mathrm{e}^{-2t}\cos(3t), \quad \mathrm{e}^{-2t}\sin(3t). \quad \boxed{2}$$

b) In order to obtain a particular solution $y_p(t)$ of the inhomogeneous equation, we solve the three equations $a(\mathrm{D})y_i = b_i(t)$ for $b_1(t) = 4$, $b_2(t) = 5\,\mathrm{e}^{2t}$, $b_3(t) = \mathrm{e}^{\mathrm{i}t}$. Superposition then yields the particular solution $y_p(t) = y_1(t) + y_2(t) - \operatorname{Im} y_3(t)$.

(1) Since $\mu = 0$ is not a root of $a(X)$, the correct Ansatz is $y_1(t) = c$, a constant. For c we obtain the equation $52\,c = 4$, so that $c = \frac{4}{52} = \frac{1}{13}$. $\boxed{1}$

(2) Since $\mu = 2$ is a root of $a(X)$ of multiplicity 2, here the correct Ansatz is $y_2(t) = c\,t^2\mathrm{e}^{2t}$.

$$\begin{aligned}
a(\mathrm{D})y_2 &= (\mathrm{D}^2 + 4\mathrm{D} + 13)(\mathrm{D} - 2)^2 \left[c\,t^2\mathrm{e}^{2t}\right] \\
&= c(\mathrm{D}^2 + 4\mathrm{D} + 13)(\mathrm{D} - 2)^2 \left[t^2\mathrm{e}^{2t}\right] \\
&= c(\mathrm{D}^2 + 4\mathrm{D} + 13)(\mathrm{D} - 2) \left[2t\,\mathrm{e}^{2t}\right] \\
&= c(\mathrm{D}^2 + 4\mathrm{D} + 13) \left[2\,\mathrm{e}^{2t}\right] \\
&= 2c(2^2 + 4 \cdot 2 + 13)\mathrm{e}^{2t} \\
&= 50\,c\,\mathrm{e}^{2t}.
\end{aligned}$$

$\Longrightarrow c = \frac{1}{10}$ $\boxed{1\frac{1}{2}}$

(3) Since $\mu = \mathrm{i}$ is not a root of $a(X)$, here the solution is $y_3(t) = c\,\mathrm{e}^{\mathrm{i}t}$ with

$$c = \frac{1}{a(\mathrm{i})} = \frac{1}{52 - 36\mathrm{i}} = \frac{1}{4(13 - 9\mathrm{i})} = \frac{13 + 9\mathrm{i}}{4(13^2 + 9^2)} = \frac{13 + 9\mathrm{i}}{1000}. \qquad \boxed{1}$$

Putting things together gives

$$\begin{aligned}
y_p(t) &= \frac{1}{13} + \frac{1}{10}\,t^2\mathrm{e}^{2t} - \operatorname{Im}\left(\frac{13 + 9\mathrm{i}}{1000}\,\mathrm{e}^{\mathrm{i}t}\right) \\
&= \frac{1}{13} + \frac{1}{10}\,t^2\mathrm{e}^{2t} - \frac{13}{1000}\,\sin t - \frac{9}{1000}\,\cos t. \qquad \boxed{\tfrac{1}{2}}
\end{aligned}$$

The general real solution is then

$$y(t) = y_p(t) + c_1\mathrm{e}^{2t}t + c_2 t\,\mathrm{e}^{2t} + c_3\mathrm{e}^{-2t}\cos(3t) + c_4\mathrm{e}^{-2t}\sin(3t), \quad c_1, c_2, c_3, c_4 \in \mathbb{R}. \qquad \boxed{1}$$

Remarks: In b) the constant solution $y_1(t) \equiv \frac{1}{13}$ was found by allmost all students. Many students had difficulties with computing the other two solutions, perhaps since they didn't know the computational trick in (2) using $(\mathrm{D} - \mu)\left[p(t)\mathrm{e}^{\mu t}\right] = p'(t)\mathrm{e}^{\mu t}$ for polynomials $p(t)$, or the method of complexification used in (3).

$$\sum_{6} = 8$$

$$\sum_{\text{Final Exam}} = 12 + 10 + 5 + 11 + 7 + 8 = 53 = 45 + 8$$
