Question 1 (ca. 12 marks)

Decide whether the following statements are true or false, and justify your answers.

a) The (maximal) solution $y(t)$ of the IVP $y' = y^{2024} - 2,\ y(0) = 0$ satisfies $y(1) = 1$.

b) The solution $y(t)$ of the IVP in a) satisfies $y(1) = -1$.

c) The solution curve of the IVP in a) in the (t, y) plane is point-symmetric to $(0, 0)$.

d) Every maximal solution of $(x^2 + 1)y'' + 2x\,y' - 6y = 0$ has domain $\mathbb{R}$.

e) The differential equation in d) has a nonzero power series solution $y(x) = \sum_{n=0}^{\infty} a_n x^n$ that is defined for $x = 2$.

f) If $\mathbf{A} \in \mathbb{R}^{n\times n}$ satisfies $\mathbf{A}^2 = \mathbf{A}$ then any system $\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{b}$, $\mathbf{b} \in \mathbb{R}^n$, has a solution of the form $\mathbf{y}(t) = \mathbf{w}_0 + t\,\mathbf{w}_1$ with $\mathbf{w}_0, \mathbf{w}_1 \in \mathbb{R}^n$.

Question 2 (ca. 10 marks)

Consider the differential equation

$$2x^2y'' + 3x(x + 1)y' - 6y = 0. \tag{DE}$$

a) Verify that $x_0 = 0$ is a regular singular point of (DE).

b) Determine the general solution of (DE) on $(0, \infty)$.

c) Using the result of b), state the general solution of (DE) on $(-\infty, 0)$ and on $\mathbb{R}$.

Question 3 (ca. 7 marks)

Consider the differential equation

$$y' = y^2 + \frac{1}{4t^2}, \qquad t > 0. \tag{R}$$

a) Show that there exists a solution $y_1(t)$ of the form $y_1(t) = c\,t^r$ with constants c, r.

b) Show that the substitution $y = y_1 + 1/z$ transforms (R) into a first-order linear equation.

c) Using b), determine all maximal solutions of (R) and their domains.

Question 4 (ca. 9 marks)

Consider $\mathbf{A} = \begin{pmatrix} -7 & -4 & 5 \\ 21 & 12 & -11 \\ 15 & 8 & -5 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$.

a) Determine a fundamental system of solutions of the system $\mathbf{y}' = \mathbf{A}\mathbf{y}$.

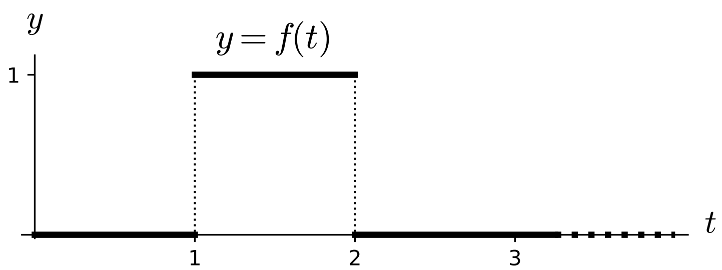
b) Solve the initial value problem $\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{b},\ \mathbf{y}(0) = (0,0,0)^\mathsf{T}$.
Hint: $\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{b}$ has a constant solution.

Question 5 (ca. 6 marks)

For the function f sketched below, solve the initial value problem

$$y'' + 5\,y' + 6y = f(t), \quad y(0) = y'(0) = 1$$

with the Laplace transform.



Notes: For the solution $y(t)$ explicit formulas valid in the intervals $[0,1]$, $[1,2]$, $[2,\infty)$ are required. You *must* use the Laplace transform for the computation.

Question 6 (ca. 8 marks)

a) Determine a real fundamental system of solutions of

$$y^{(4)} - 7\,y'' + 4\,y' + 20\,y = 0.$$

b) Determine the general real solution of

$$y^{(4)} - 7\,y'' + 4\,y' + 20\,y = \mathrm{e}^{-2t}(1 - 8\sin t).$$

Solutions

1 a) False. The function $f(y)=y^{2024}-2$ has exactly two real zeros $z_1=-\sqrt[2024]{2}$, which is slightly smaller than -1, and $z_2=\sqrt[2024]{2}$, which is slightly larger than 1. Since $f(y)<0$ for $y\in(z_1,z_2)$ and $y(0)=0\in(z_1,z_2)$, the solution $y(t)$ must be decreasing and hence cannot have $y(1)>0$. $\boxed{2}$

b) False. Integrating the ODE gives

$$\int_{-1}^{0}\frac{\mathrm{d}y}{y^{2024}-2}=-\int_{0}^{-1}\frac{\mathrm{d}y}{y^{2024}-2}=-\int_{y(0)}^{y(1)}\frac{\mathrm{d}y}{y^{2024}-2}=-\int_0^1\mathrm{d}t=-1.$$

The integrand $g(y)=\frac{1}{y^{2024}-2}$ is continuous and strictly decreasing with $g(0)=-1/2$, $g(1)=-1$. This implies $\int_0^1 g(y)\,\mathrm{d}y>-1$, contradiction. $\boxed{2}$

c) True. Consider $z(t)=-y(-t)$, $t\in(z_1,z_2)$. (For the meaning of z_1,z_2 see the solution to a).) We have $z(0)=-y(0)=0$ and $z'(t)=y'(-t)=y(-t)^{2024}-2=\bigl(-y(-t)\bigr)^{2024}-2=z(t)^{2024}-2$. Thus $z(t)$ solves the same IVP as $y(t)$. By the Uniqueness Theorem we must have $z(t)=y(t)$ for $t\in(z_1,z_2)$, which proves the assertion. $\boxed{2}$

d) True. The ODE is linear, and its explicit form is $y''+\frac{2x}{x^2+1}\,y'-\frac{6}{x^2+1}\,y=0$. Since the coefficient functions have maximal domain $\mathbb{R}$, the same is true of every maximal solution (by the the sharpened version of the Existence Uniqueness Theorem for linear ODEs). $\boxed{2}$

e) True (surprise?). The ODE has the polynomial solution $y(x)=3\,x^2+1$, which is a power series of the required form (with $a_0=1$, $a_2=3$ and all other a_n equal to zero). $\boxed{2}$

For discovering this solution no ingenuity is required. Putting the power series Ansatz into the ODE gives

$$\sum_{n=0}^{\infty}\Bigl[\bigl(n(n-1)+2n-6\bigr)a_n+(n+1)(n+2)a_{n+2}\Bigr]x^n=0.$$

Hence the coefficients satisfy the recurrence relation

$$a_{n+2}=-\frac{n^2+n-6}{(n+1)(n+2)}\,a_n=-\frac{(n-2)(n+3)}{(n+1)(n+2)}\,a_n,\qquad n=0,1,2,\dots$$

Setting $(a_0,a_1)=(1,0)$ we obtain $a_n=0$ for odd n, $a_2=-\frac{(0-2)(0+3)}{(0+1)(0+2)}=3$, $a_4=-\frac{(2-2)(2+3)}{(2+1)(2+2)}=0$, and by induction $a_n=0$ for all even $n\geq 4$. Thus $\sum_{n=0}^{\infty}a_nx^n=3\,x^2+1$.

f) True. For $\mathbf{y}(t)=\mathbf{w}_0+t\,\mathbf{w}_1$ we have

$$\mathbf{w}_1=\mathbf{y}'=\mathbf{A}\mathbf{y}+\mathbf{b}=\mathbf{A}\left(\mathbf{w}_0+t\,\mathbf{w}_1\right)+\mathbf{b}\quad\Longleftrightarrow\quad\mathbf{w}_1=\mathbf{A}\mathbf{w}_0+\mathbf{b}\wedge\mathbf{A}\mathbf{w}_1=\mathbf{0}.$$

The latter system has a solution iff there exists $\mathbf{w}_0\in\mathbb{R}^n$ such that $\mathbf{A}\left(\mathbf{A}\mathbf{w}_0+\mathbf{b}\right)=\mathbf{0}$. Since

$$\mathbf{A}\left(\mathbf{A}\mathbf{w}_0+\mathbf{b}\right)=\mathbf{A}^2\mathbf{w}_0+\mathbf{A}\mathbf{b}=\mathbf{A}\mathbf{w}_0+\mathbf{A}\mathbf{b}=\mathbf{A}(\mathbf{w}_0+\mathbf{b}),$$

we can choose $\mathbf{w}_0 = -\mathbf{b}$, i.e., $\mathbf{y}(t) = -\mathbf{b} + t\big(\mathbf{A}(-\mathbf{b}) + \mathbf{b}\big) = -\mathbf{b} + t(\mathbf{I} - \mathbf{A})\mathbf{b}$. [2]

Remarks: No marks were assigned for answers without justification.

a) This question was answered correctly by most students. Pictures showing the approximate shape of the solution curve were accepted. If "$y'(t) \in (-2,-1)$ for $t \in (0,1)$" (instead of "$y \in (0,1)$") was stated without justification, I have subtracted 0.5 marks.

A few students claimed that $\sqrt[2024]{2} \in (0,1)$ and that the constant solution $y(t) \equiv \sqrt[2024]{2}$ prevents $y(t)$ from attaining the value 1, which is of course false.

b) This question was answered correctly by many students, mostly using an argument like "if true, the average slope of $y(t)$ in $[0,1]$ would be -1 but $y'(t)$ varies between $y'(0) = -2$ and $y'(1) = -1$." (Note that this argument also relies on the fact that $y(t)$ is decreasing in $[0,1]$; cf. a).) If one doesn't assume $y(1) = -1$, the slope argument gives $y(1) < -1$. But also many students failed to produce a correct proof or even answered "True", probably because the solution attains the value $y(t_1) = -1$ at some point t_1. However, t_1 is not 1 but smaller than 1.

No marks were assigned for observing that if one of a), b) is True then the other must be False.

c) This was admittedly the most difficult question of the exam. But in the homework we had an exercise about the logistic equation, where such a symmetry had to be shown and the suggested solution method was given as a hint. Only a handful of students solved this problem. Most students claimed that $y'(t)$ must be an even function (because the exponent 2024 is even), but this is not obvious. In fact, $y(t)^{2024} = y(-t)^{2024}$ can only hold if $y(-t) = \pm y(t)$ and thus is merely a restatement of the result to be proven. If one considers t as function of y, however, one can put this argument to work. Using the explicit formula $y(t) = y(0) + \int_0^t y'(s)\,\mathrm{d}s = \int_0^t y(s)^{2024} - 2\,\mathrm{d}s$ together with the substitution $s = -t$ also doesn't work, because from "$y(-s) = -y(s)$ for all $s < t$ implies $y(-t) = -y(t)$" one cannot conclude the required symmetry property. (There is no mathematical induction over the positive reals, because they are not well-ordered.)

d) Many students answered this question correctly, but also many confused it with a problem about analytic ODE's. It is correct that the nonexistence of a singular point (in $\mathbb{R}$) implies that maximal solutions have domain $\mathbb{R}$, but the reason for this is that the ODE is linear. It can't be proved by power series solutions, some of which have only a finite radius of convergence (due to the singularities at $\pm\mathrm{i}$). If students argued with analytic solutions but"linearity" wasn't mentioned, I have subtracted 0.5 marks.

e) Only 1 student (if I remember correctly) found the polynomial solution, obtaining 2 marks for this question. Partial credit for observing that the guaranteed radius of convergence is only 1 wasn't given this time, because I followed strictly the rule that wrong answers receive 0 marks. Some students didn't notice that the power series solution is required to have center $x_0 = 0$ and used $x_0 = 2$ instead. Subject to this they answered correctly "True". But this couldn't be honored by any marks, of course.

f) Only few students answered this question completely. Many stopped short of completing the argument after finding the condition $\mathbf{w}_1 = \mathbf{A}\mathbf{w}_0 + \mathbf{b} \wedge \mathbf{A}\mathbf{w}_1 = \mathbf{0}$. I have assigned 1 mark in such cases. An error that was occasionally made is the false implication "$\mathbf{A}^2 = \mathbf{A} \Longrightarrow \mathbf{A} = \mathbf{0} \vee \mathbf{A} = \mathbf{I}$". (Projection matrices wouldn't be interesting if that were True.) This error is also implicit in the conclusion that $\mathbf{A}(\mathbf{w}_0 + \mathbf{b}) = \mathbf{0}$ implies $\mathbf{w}_0 = -\mathbf{b}$.

$$\sum_1 = 12$$

2 a) The explicit form of (DE) is

$$y'' + \frac{3(x+1)}{2x}\, y' - \frac{3}{x^2}\, y = 0$$

$p(x) := \frac{3}{2}\frac{1}{x} + \frac{3}{2}$ has a pole of order 1 at 0, and $q(x) := -\frac{3}{x^2}$ has a pole of order 2 at 0. This shows that 0 is a regular singular point of (DE). [1]

Alternatively, use that the limits defining p_0, q_0 below are finite.

b) From a) we have $p_0 = \lim_{x\to 0} x\, p(x) = \frac{3}{2}$, $q_0 = \lim_{x\to 0} x^2\, q(x) = -3$. (These coefficients can just be read off from the explicit form.)
$\Longrightarrow$ The indicial equation is

$$r^2 + (p_0 - 1)r + q_0 = r^2 + \frac{1}{2}r - 3 = \left(r - \frac{3}{2}\right))(r+2) = 0.$$

$\Longrightarrow$ The exponents at the singularity $x_0 = 0$ are $r_1 = 3/2$, $r_2 = -2$. [1]

Since $r_1 - r_2 \notin \mathbb{Z}$, there exist two fundamental solutions y_1, y_2 of the form

$$y_1(x) = x^{3/2}\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+3/2},$$
$$y_2(x) = x^{-2}\sum_{n=0}^{\infty} b_n x^n = \sum_{n=0}^{\infty} b_n x^{n-2}$$

with normalization $a_0 = b_0 = 1$.

First we determine $y_1(x)$. We have

$$\begin{aligned}
0 &= 2\,x^2 y_1'' + 3x(x+1)y_1' - 6\,y_1 \\
&= 2\,x^2\sum_{n=0}^{\infty}(n+3/2)(n+1/2)a_n x^{n-1/2} + (3\,x^2 + 3x)\sum_{n=0}^{\infty}(n+3/2)a_n x^{n+1/2} - 6\sum_{n=0}^{\infty} a_n x^{n+3/2} \\
&= 2\sum_{n=0}^{\infty}(n+3/2)(n+1/2)a_n x^{n+3/2} + 3\sum_{n=0}^{\infty}(n+3/2)a_n x^{n+5/2} + 3\sum_{n=0}^{\infty}(n+3/2)a_n x^{n+3/2} - 6\sum_{n=0}^{\infty} a_n x^{n+3/2} \\
&= 2\sum_{n=0}^{\infty}(n+3/2)(n+1/2)a_n x^{n+3/2} + 3\sum_{n=1}^{\infty}(n+1/2)a_{n-1} x^{n+3/2} + 3\sum_{n=0}^{\infty}(n+3/2)a_n x^{n+3/2} - 6\sum_{n=0}^{\infty} a_n x^{n+3/2} \\
&= 0\,a_0 + \sum_{n=1}^{\infty}\left[\left(2n^2 + 4n + 3/2 + 3n + 9/2 - 6\right)a_n + 3(n+1/2)a_{n-1}\right]x^{n+3/2} \\
&= \sum_{n=1}^{\infty}\left[\left(2n^2 + 7n\right)a_n + 3(n+1/2)a_{n-1}\right]x^{n+3/2}.
\end{aligned}$$

Equating coefficients gives the recurrence relation

$$a_n = -\frac{3(n+1/2)}{2n^2+7n}\,a_{n-1} = \frac{3(2n+1)}{2n(2n+7)}\,a_{n-1} \quad \text{for } n = 1,2,3,\dots, \qquad \boxed{1}$$

and with $a_0 = 1$ further $a_n = (-1)^n \frac{3^n \cdot 3 \cdot 5 \cdots (2n+1)}{2\cdot 4 \cdots (2n)\cdot 9 \cdot 11 \cdots (2n+7)}$ for $n \geq 1$. $\boxed{1}$

$$\implies y_1(x) = x^{3/2} - \frac{3\cdot 3}{2\cdot 9}\,x^{5/2} + \frac{3^2\cdot 3\cdot 5}{2\cdot 4\cdot 9\cdot 11}\,x^{7/2} - \frac{3^3\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 9\cdot 11\cdot 13}\,x^{9/2} + \frac{3^4\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 11\cdot 13\cdot 15}\,x^{11/2} + \cdots$$
$$= x^{3/2} - \frac{1}{2}\,x^{5/2} + \frac{15}{88}\,x^{7/2} + \frac{35}{16}\sum_{n=3}^{\infty} \frac{(-1)^n 3^n}{(2n+3)(2n+5)(2n+7)}\,x^{n+3/2}. \qquad \boxed{\tfrac{1}{2}}$$

For the determination of $y_2(x)$ we repeat the process with exponents decreased by 3.5 :

$$\begin{aligned}
0 &= 2\,x^2 y_2'' + 3x(x+1)y_2' - 6\,y_2 \\
&= 2\,x^2 \sum_{n=0}^{\infty}(n-2)(n-3)b_n x^{n-4} + (3\,x^2+3x)\sum_{n=0}^{\infty}(n-2)b_n x^{n-3} - 6\sum_{n=0}^{\infty} b_n x^{n-2} \\
&= 2\sum_{n=0}^{\infty}(n-2)(n-3)b_n x^{n-2} + 3\sum_{n=0}^{\infty}(n-2)b_n x^{n-1} + 3\sum_{n=0}^{\infty}(n-2)b_n x^{n-2} - 6\sum_{n=0}^{\infty} b_n x^{n-2} \\
&= 2\sum_{n=0}^{\infty}(n-2)(n-3)b_n x^{n-2} + 3\sum_{n=1}^{\infty}(n-3)b_{n-1} x^{n-2} + 3\sum_{n=0}^{\infty}(n-2)b_n x^{n-2} - 6\sum_{n=0}^{\infty} b_n x^{n-2} \\
&= 0\,b_0 + \sum_{n=1}^{\infty}\left[\left(2n^2 - 10n + 12 + 3n - 6 - 6\right)b_n + 3(n-3)b_{n-1}\right]x^{n-2} \\
&= \sum_{n=1}^{\infty}\left[\left(2n^2 - 7n\right)b_n + 3(n-3)b_{n-1}\right]x^{n-2}.
\end{aligned}$$

Here we obtain the recurrence relation

$$b_n = -\frac{3(n-3)}{n(2n-7)}\,b_{n-1} \quad \text{for } n = 1,2,3,\dots. \qquad \boxed{1}$$

Setting $b_0 = 1$ gives $b_n = (-1)^n \frac{3^n\cdot(-2)(-1)\cdots(n-3)}{n!(-5)(-3)\cdots(2n-7)}$ for $n \geq 1$, i.e., $b_1 = -6/5$, $b_2 = 3/5$ and $b_n = 0$ for all $n \geq 3$.

$$\implies y_2(x) = x^{-2} - \frac{6}{5}\,x^{-1} + \frac{3}{5}. \qquad \boxed{1\tfrac{1}{2}}$$

Alternative solution: We use the general recurrence relation for the rational functions $a_n(r)$, viz. $a_0(r) = 1$ and

$$a_n(r) = -\frac{1}{F(r+n)}\sum_{k=0}^{n-1}\left[(r+k)p_{n-k} + q_{n-k}\right]a_k(r) \quad \text{for } n \geq 1.$$

Since $F(r) = (r-3/2)(r+2)$ and all coefficients p_i, q_i except for $p_0 = p_1 = 3/2$,

$q_0 = -3$ are zero, we obtain

$$\begin{aligned} a_n(r) &= -\frac{(r+n-1)(3/2)}{(r+n-3/2)(r+n+2)}\, a_{n-1}(r) \\ &= -\frac{3(r+n-1)}{(2r+2n-3)(r+n+2)}\, a_{n-1}(r) \\ &= \dots = \frac{(-1)^n 3^n r(r+1)\cdots(r+n-1)}{(2r-1)(2r+1)\cdots(2r+2n-3)(r+3)(r+4)\cdots(r+n+2)} \end{aligned}$$

(valid for $r \notin \{1/2, -1/2, -3/2, \dots\}$ and $r \notin \{-3, -4, -5, \dots\}$). For $n \geq 4$ this simplifies to

$$a_n(r) = \frac{(-1)^n 3^n r(r+1)(r+2)}{(2r-1)(2r+1)\cdots(2r+2n-3)(r+n)(r+n+1)(r+n+2)}.$$

Substituting $r_1 = 3/2$, $r_2 = -2$ into the recurrence relation for $a_n(r)$ gives the recurrence relations for a_n, b_n obtained above. (In terms of the functions $a_n(r)$ these numbers are $a_n(3/2)$ and $a_n(-2)$, respectively.) The explicit form of a_n, b_n can be obtained from the explicit form of the functions $a_n(r)$ in the same way.

The general (real) solution on $(0, \infty)$ is then $y(x) = c_1\, y_1(x) + c_2\, y_2(x)$, $c_1, c_2 \in \mathbb{R}$. $\boxed{\frac{1}{2}}$

That solutions are defined on the whole of $(0, \infty)$, is guaranteed by the analyticity of $p(x)$, $q(x)$ in $\mathbb{C} \setminus \{0\}$, but follows also readily from the easily established fact that the radius of convergence of $\sum_{n=0}^{\infty} a_n x^n$ is ∞. $\boxed{\frac{1}{2}}$

c) The solution on $(-\infty, 0)$ is $y(x) = c_1\, y_1^-(x) + c_2\, y_2(x)$ with the same function $y_2(x)$ as in b) and

$$y_1^-(x) = (-x)^{3/2} \sum_{n=0}^{\infty} a_n x^n = |x|^{3/2} \sum_{n=0}^{\infty} a_n x^n. \qquad \boxed{1}$$

(The functions $y_1(-x)$ and $y_2(-x)$ are defined on $(-\infty, 0)$, but they are not solutions!)

Since $y_1(x) \simeq x^{3/2}$ for $x \downarrow 0$, the (unique) continuous extension of $y_1(x)$ to $[0, \infty)$ (obtained by setting $y_1(0) = 0$) is not twice differentiable at $x = 0$ (only once differentiable). Hence it cannot be a solution on $[0, \infty)$. The same argument works for any constant multiple of $y_1(x)$, showing that the only solution on $[0, \infty)$, and hence on $\mathbb{R}$, is $y(x) \equiv 0$. $\boxed{1}$

Remarks: In b) some students obtained wrong singularity exponents r_1, r_2, which invalidates almost all remaining computations. If that error has happend, the only way to cure this problem is to note when equating coefficients that the first equation implies $a_0(r_i) = 0$ for those r_i, which shows that something is wrong, and then redo the computation of r_i.

In b), as every year, quite a few students copy & pasted the sentence “That solutions are defined on the whole of $(0, \infty)$ …” into the exam paper. Of course this gives the final 0.5 marks for b), but it is a waste of time. For example, you could just use the first half of it, and in abbreviated form.

In c) many students got the solution on $(-\infty, 0)$ wrong (either by using $y_1(-x)$ and $y_2(-x)$, which is wrong in both cases, or by not specifying the meaning of $y_1^-(x)$). Many more claimed that the solution on $\mathbb{R}$ is all constant multiples of

$$x \mapsto \begin{cases} y_1(x) & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ y_1^-(x) & \text{if } x < 0, \end{cases}$$

which is false as explained above.

$$\sum_2 = 10$$

3 a) Substituting $y_1(t) = c\,t^r$ into the ODE gives

$$cr\,t^{r-1} = c^2 t^{2r} + \frac{1}{4}t^{-2},$$

which holds if $r = -1$ and $-c = c^2 + \frac{1}{4}$, i.e., $4c^2 + 4c + 1 = 0$, which has the solution $c = -\frac{1}{2}$. Thus we can take $y_1(t) = -\frac{1}{2t}$. [1]

b) Substituting $y = y_1(t) + \frac{1}{z} = -\frac{1}{2t} + \frac{1}{z}$ in (R) we obtain

$$\begin{aligned} \frac{1}{2t^2} - \frac{z'}{z^2} &= \left(-\frac{1}{2t} + \frac{1}{z}\right)^2 + \frac{1}{4t^2} = \frac{1}{4t^2} - \frac{1}{tz} + \frac{1}{z^2} + \frac{1}{4t^2} \\ \iff \quad -\frac{z'}{z^2} &= -\frac{1}{tz} + \frac{1}{z^2} \\ \iff \quad z' &= \frac{1}{t}\,z - 1 \end{aligned}$$

[2]

c) The general solution of $z' = (1/t)z$ is

$$z(t) = c\exp\int \frac{1}{t}\,\mathrm{d}t = ct, \quad c \in \mathbb{R}.$$ [1]

Variation of parameters then yields a particular solution z_p of $z' = (1/t)z - 1$:

$$z_p(t) = t\int \frac{1}{t}(-1)\,\mathrm{d}t = -t\ln t.$$ [1]

$\Longrightarrow$ The general solution of $z' = (1/t)z - 1$ is

$$z(t) = ct - t\ln t, \quad c \in \mathbb{R}.$$

$\Longrightarrow$ The general solution of (R) is

$$y(t) = -\frac{1}{2t} + \frac{1}{ct - t\ln t} = \frac{1}{t}\left(\frac{1}{c - \ln t} - \frac{1}{2}\right), \quad c \in \mathbb{R} \cup \{\infty\},$$ [1]

where $c = \infty$ represents the solution y_1.

The maximal domain of y_1 is $(0, \infty)$. For $c \in \mathbb{R}$ the expression for $y(t)$ defines two maximal solutions $y_c^{(1)}$ with domain $(0, \mathrm{e}^c)$ and $y_c^{(2)}$ with domain (e^c, ∞). [1]

Remarks: Many students had problems with this question, which is considered fairly standard (and similar problems have been discussed in the homework). Most students were able to solve a), but then stopped short of finding the first-order linear equation for $z(t)$ in b). In c) I noted that even of those who solved a) and b) correctly quite a few couldn't identify the corresponding maximal domains, which must be intervals!

$$\sum_3 = 7$$

4 a) The characteristic polynomial of $\mathbf{A}$ is

$$\chi_{\mathbf{A}}(X) = (-1)^3 \begin{vmatrix} -7-X & -4 & 5 \\ 21 & 12-X & -11 \\ 15 & 8 & -5-X \end{vmatrix} = -\begin{vmatrix} -7-X & -4 & 5 \\ -3X & -X & 4 \\ 1-2X & 0 & 5-X \end{vmatrix}$$

$$= -\begin{vmatrix} 5-X & -4 & 5 \\ 0 & -X & 4 \\ 1-2X & 0 & 5-X \end{vmatrix} = \begin{vmatrix} 5-X & -4 & 5 \\ 0 & X & -4 \\ 1-2X & 0 & 5-X \end{vmatrix}$$
$$= X(X-5)^2 + 16(1-2X) - 5X(1-2X)$$
$$= X^3 - 12X + 16$$
$$= (X-2)(X^2+2X-8)$$
$$= (X-2)^2(X+4).$$

$\Longrightarrow$ The eigenvalues of $\mathbf{A}$ are $\lambda_1 = 2$ with algebraic multiplicity 2 and $\lambda_2 = -4$ with algebraic multiplicity 1. $\boxed{2}$

$$\mathbf{A}-2\mathbf{I} = \begin{pmatrix} -9 & -4 & 5 \\ 21 & 10 & -11 \\ 15 & 8 & -7 \end{pmatrix} \to \begin{pmatrix} -9 & -4 & 5 \\ 3 & 2 & -1 \\ 15 & 8 & -7 \end{pmatrix} \to \begin{pmatrix} 3 & 2 & -1 \\ 0 & 2 & 2 \\ 0 & -2 & -2 \end{pmatrix} \to \begin{pmatrix} 3 & 2 & -1 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

$\Longrightarrow$ The eigenspace corresponding to $\lambda_1 = 2$ is generated by $\mathbf{v}_1 = (1,-1,1)^\mathsf{T}$, and $\mathbf{A}$ is not diagonalisable.

A further generalized eigenvector $\mathbf{v}_2$ corresponding to $\lambda_1 = 2$ is obtained by solving $(\mathbf{A} - 2\mathbf{I})\mathbf{x} = \mathbf{v}_1$.

$$\left(\begin{array}{ccc|c} -9 & -4 & 5 & 1 \\ 21 & 10 & -11 & -1 \\ 15 & 8 & -7 & 1 \end{array}\right) \to \left(\begin{array}{ccc|c} -9 & -4 & 5 & 1 \\ 3 & 2 & -1 & 1 \\ 15 & 8 & -7 & 1 \end{array}\right) \to \left(\begin{array}{ccc|c} 3 & 2 & -1 & 1 \\ 0 & 2 & 2 & 4 \\ 0 & -2 & -2 & -4 \end{array}\right) \to \left(\begin{array}{ccc|c} 3 & 2 & -1 & 1 \\ 0 & 2 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$\Longrightarrow$ We can take $\mathbf{v}_2 = (0,1,1)^\mathsf{T}$.

$$\mathbf{A}+4\mathbf{I} = \begin{pmatrix} -3 & -4 & 5 \\ 21 & 16 & -11 \\ 15 & 8 & -1 \end{pmatrix} \to \begin{pmatrix} -3 & -4 & 5 \\ 0 & -12 & 24 \\ 0 & -12 & 24 \end{pmatrix} \to \begin{pmatrix} -3 & -4 & 5 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

$\Longrightarrow$ The eigenspace corresponding to $\lambda_3 = -4$ is generated by $\mathbf{v}_3 = (-1,2,1)^\mathsf{T}$

A real fundamental system of solutions is then

$$\mathbf{y}_1(t) = \mathrm{e}^{2t}\mathbf{v}_1 = \mathrm{e}^{2t}\begin{pmatrix}1\\-1\\1\end{pmatrix}, \qquad \boxed{1}$$

$$\mathbf{y}_2(t) = \mathrm{e}^{2t}\mathbf{v}_2 + t\,\mathrm{e}^{2t}\mathbf{v}_1 = \mathrm{e}^{2t}\begin{pmatrix}0\\1\\1\end{pmatrix} + t\,\mathrm{e}^{2t}\begin{pmatrix}1\\-1\\1\end{pmatrix}, \qquad \boxed{2}$$

$$\mathbf{y}_3(t) = \mathrm{e}^{-4t}\mathbf{v}_3 = \mathrm{e}^{-4t}\begin{pmatrix}-1\\2\\1\end{pmatrix}. \qquad \boxed{1}$$

b) Since $\mathbf{A}$ is invertible, $\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{b}$ has the constant solution $\mathbf{y}(t) \equiv -\mathbf{A}^{-1}\mathbf{b}$, which is obtained by solving $\mathbf{A}\mathbf{x} = -\mathbf{b}$.

$$\left(\begin{array}{rrr|r} -7 & -4 & 5 & 0\\ 21 & 12 & -11 & -1\\ 15 & 8 & -5 & -1 \end{array}\right) \to \left(\begin{array}{rrr|r} -7 & -4 & 5 & 0\\ 0 & 0 & 4 & -1\\ 1 & 0 & 5 & -1 \end{array}\right)$$

$\Longrightarrow x_3 = -1/4$, $x_1 = -1 - 5\,x_3 = 1/4$, $x_2 = (-7\,x_1 + 5\,x_3)/4 = -3/4$, so that

$$\mathbf{y}_p(t) \equiv \frac{1}{4}\begin{pmatrix}1\\-3\\-1\end{pmatrix} \quad \text{is a particular solution.} \qquad \boxed{1}$$

The general solution of $\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{b}(t)$ is $\mathbf{y}(t) = \mathbf{y}_p(t) + c_1\mathbf{y}_1(t) + c_2\mathbf{y}_2(t) + c_3\mathbf{y}_3(t)$. In order to satisfy the required initial condition, (c_1, c_2, c_3) needs to solve

$$\left(\begin{array}{rrr|r} 1 & 0 & -1 & -\frac{1}{4}\\ -1 & 1 & 2 & \frac{3}{4}\\ 1 & 1 & 1 & \frac{1}{4} \end{array}\right) \to \left(\begin{array}{rrr|r} 1 & 0 & -1 & -\frac{1}{4}\\ 0 & 1 & 1 & \frac{1}{2}\\ 0 & 1 & 2 & \frac{1}{2} \end{array}\right) \to \left(\begin{array}{rrr|r} 1 & 0 & -1 & -\frac{1}{4}\\ 0 & 1 & 1 & \frac{1}{2}\\ 0 & 0 & 1 & 0 \end{array}\right)$$

$\Longrightarrow c_1 = -1/4$, $c_2 = 1/2$, $c_3 = 0$, $\boxed{2}$
and the final answer is

$$y(t) = \frac{1}{4}\begin{pmatrix}1\\-3\\-1\end{pmatrix} - \frac{1}{4}\mathrm{e}^{2t}\begin{pmatrix}1\\-1\\1\end{pmatrix} + \frac{1}{2}\left(\mathrm{e}^{2t}\begin{pmatrix}0\\1\\1\end{pmatrix} + t\,\mathrm{e}^{2t}\begin{pmatrix}1\\-1\\1\end{pmatrix}\right)$$
$$= \begin{pmatrix} \frac{1}{4} - \frac{1}{4}\mathrm{e}^{2t} + \frac{1}{2}t\,\mathrm{e}^{2t}\\ -\frac{3}{4} + \frac{3}{4}\mathrm{e}^{2t} - \frac{1}{2}t\,\mathrm{e}^{2t}\\ -\frac{1}{4} + \frac{1}{4}\mathrm{e}^{2t} + \frac{1}{2}t\,\mathrm{e}^{2t} \end{pmatrix}.$$

Remarks: a) was solved by many students. Many others didn't know enough about generalized eigenvectors to produce a 3rd fundamental solution. Those who attended Lectures 37 and 38 had an advantage, because this year we have discussed an example of exactly the same type (with $\mathbf{A}$ nondiagonalizable and having two distinct eigenvalues); see the `lecture37-38_handout.pdf`, Slides 46–48.

b) was solved by considerably fewer students. Some students couldn't make sense of the hint given in the statement of b) and tried the more general Ansatz $\mathbf{y}(t) = \mathbf{w}_0 + t\,\mathbf{w}_1$ (which nevertheless should have produced the constant solution as well!). But perhaps the most frequent problem was lack of time at the end of the examination.

$$\sum_4 = 9$$

5 Writing $Y(s) = \mathcal{L}\{y(t)\}$, $F(s) = \mathcal{L}\{f(t)\}$, and applying the Laplace transform to both sides of the ODE gives

$$\begin{aligned}\mathcal{L}\{y'' + 5\,y' + 6\,y\} &= s^2\,Y(s) - s\,y(0) - y'(0) + 5\big(s\,Y(s) - y(0)\big) + 6\,Y(s)\\ &= (s^2 + 5s + 6)Y(s) - s - 6 = \mathcal{L}\big\{f(t)\big\} = F(s).\end{aligned}$$

Further we have

$$\begin{aligned}f(t) &= \mathrm{u}_1(t) - \mathrm{u}_2(t)\\ \implies F(s) &= \frac{\mathrm{e}^{-s} - \mathrm{e}^{-2s}}{s}. && \boxed{1}\\ \implies Y(s) &= \frac{F(s) + s + 6}{s^2 + 5s + 6} = \frac{\mathrm{e}^{-s} - \mathrm{e}^{-2s}}{s(s+2)(s+3)} + \frac{s+6}{(s+2)(s+3)} && \boxed{1}\end{aligned}$$

Together with

$$\begin{aligned}\frac{1}{s(s+2)(s+3)} &= \frac{1}{6s} - \frac{1}{2(s+2)} + \frac{1}{3(s+3)},\\ \frac{s+6}{(s+2)(s+3)} &= \frac{4}{s+2} - \frac{3}{s+3}, && \boxed{1}\\ \mathcal{L}\left\{\frac{1}{6} - \frac{1}{2}\,\mathrm{e}^{-2t} + \frac{1}{3}\,\mathrm{e}^{-3t}\right\} &= \frac{1}{6s} - \frac{1}{2(s+2)} + \frac{1}{3(s+3)},\\ \mathcal{L}\left\{4\,\mathrm{e}^{-2t} - 3\,\mathrm{e}^{-3t}\right\} &= \frac{4}{s+2} - \frac{3}{s+3} && \boxed{1}\end{aligned}$$

this gives

$$\begin{aligned}y(t) &= 4\,\mathrm{e}^{-2t} - 3\,\mathrm{e}^{-3t}\\ &\quad + \mathrm{u}_1(t)\left(\frac{1}{6} - \frac{1}{2}\,\mathrm{e}^{-2(t-1)} + \frac{1}{3}\,\mathrm{e}^{-3(t-1)}\right)\\ &\quad - \mathrm{u}_2(t)\left(\frac{1}{6} - \frac{1}{2}\,\mathrm{e}^{-2(t-2)} + \frac{1}{3}\,\mathrm{e}^{-3(t-2)}\right) && \boxed{1}\\ &= \begin{cases} 4\,\mathrm{e}^{-2t} - 3\,\mathrm{e}^{-3t} & \text{for } 0 \le t \le 1,\\ \frac{1}{6} + \left(4 - \frac{1}{2}\,\mathrm{e}^2\right)\mathrm{e}^{-2t} - \left(3 - \frac{1}{3}\,\mathrm{e}^3\right)\mathrm{e}^{-3t} & \text{for } 1 \le t \le 2,\\ \left(4 - \frac{1}{2}\,\mathrm{e}^2 + \frac{1}{2}\,\mathrm{e}^4\right)\mathrm{e}^{-2t} - \left(3 - \frac{1}{3}\,\mathrm{e}^3 + \frac{1}{3}\,\mathrm{e}^6\right)\mathrm{e}^{-3t} & \text{for } t \ge 2.\end{cases} && \boxed{1}\end{aligned}$$

Remarks: This question was easier than those involving the Laplace transform in previous semesters. Nevertheless many students had problems with it—at least in the final step of producing the case-by-case definition of $y(t)$ errors jumped in frequently.

$$\sum_5 = 6$$

6 a) The characteristic polynomial is

$$\begin{aligned} a(X) &= X^4 - 7\,X^2 + 4\,X + 20 \\ &= (X+2)(X^3 - 2\,X^2 - 3X + 10) \\ &= (X+2)^2(X^2 - 4\,X + 5) \\ &= (X+2)^2(X - 2 - \mathrm{i})(X - 2 + \mathrm{i}) \end{aligned} \qquad \boxed{1}$$

with zeros $\lambda_1 = -2$ of multiplicity 2 and $\lambda_{2/3} = 2 \pm \mathrm{i}$ of multiplicity 1.

$\Longrightarrow$ A complex fundamental system of solutions is e^{-2t}, $t\,\mathrm{e}^{-2t}$, $\mathrm{e}^{(2+\mathrm{i})t}$, $\mathrm{e}^{(2-\mathrm{i})t}$, and the corresponding real fundamental system is

$$\mathrm{e}^{-2t}, \quad t\,\mathrm{e}^{-2t}, \quad \mathrm{e}^{2t}\cos(t), \quad \mathrm{e}^{2t}\sin(t). \qquad \boxed{2}$$

b) In order to obtain a particular solution $y_p(t)$ of the inhomogeneous equation, we solve the two equations $a(\mathrm{D})y_i = b_i(t)$ for $b_1(t) = \mathrm{e}^{-2t}$, $b_2(t) = \mathrm{e}^{(-2+\mathrm{i})t}$. Superposition then yields the particular solution $y_p(t) = y_1(t) - 8\,\mathrm{Im}\,y_2(t)$.

(1) Since $\mu = -2$ is a root of $a(X)$ of multiplicity 2, the correct Ansatz is $y_1(t) = c\,t^2\mathrm{e}^{-2t}$ with c a constant. Since

$$\begin{aligned} a(\mathrm{D})\left[t^2\mathrm{e}^{-2t}\right] &= (\mathrm{D}^2 - 4\mathrm{D} + 5)(\mathrm{D}+2)^2\left[t^2\mathrm{e}^{-2t}\right] \\ &= (\mathrm{D}^2 - 4\mathrm{D} + 5)\left[2\mathrm{e}^{-2t}\right] \\ &= 2\big((-2)^2 - 4(-2) + 5\big)\mathrm{e}^{-2t} = 34\,\mathrm{e}^{-2t}, \end{aligned}$$

we obtain $c = \frac{1}{34}$, $y_1(t) = \frac{1}{34}\,t^2\mathrm{e}^{-2t}$. $\boxed{1\tfrac{1}{2}}$

(2) Since $\mu = -2+\mathrm{i}$ is not a root of $a(X)$, here the correct Ansatz is $y_2(t) = c\,\mathrm{e}^{(-2+\mathrm{i})t}$. Since $a(\mathrm{D})\left[\mathrm{e}^{(-2+\mathrm{i})t}\right] = a(-2+\mathrm{i})\mathrm{e}^{(-2+\mathrm{i})t}$, we must take

$$c = \frac{1}{a(-2+\mathrm{i})} = \frac{1}{\mathrm{i}^2(-4)(-4+2\mathrm{i})} = \frac{1}{-16+8\mathrm{i}} = \frac{1}{8}\,\frac{1}{-2+\mathrm{i}} = \frac{1}{8}\,\frac{-2-\mathrm{i}}{5} = -\frac{2+\mathrm{i}}{8\cdot 5},$$

$y_2(t) = -\frac{2+\mathrm{i}}{8\cdot 5}\,\mathrm{e}^{(-2+\mathrm{i})t}$. $\boxed{1\tfrac{1}{2}}$

Putting things together gives

$$\begin{aligned} y_p(t) &= \frac{1}{34}\,t^2\mathrm{e}^{-2t} + \mathrm{Im}\left(\frac{2+\mathrm{i}}{5}\,\mathrm{e}^{(-2+\mathrm{i})t}\right) \\ &= \frac{1}{34}\,t^2\mathrm{e}^{-2t} + \frac{2}{5}\,\mathrm{e}^{-2t}\sin t + \frac{1}{5}\,\mathrm{e}^{-2t}\cos t. \end{aligned} \qquad \boxed{1}$$

The general real solution is then

$$y(t) = y_p(t) + c_1\mathrm{e}^{-2t}t + c_2t\,\mathrm{e}^{-2t} + c_3\mathrm{e}^{2t}\cos(t) + c_4\mathrm{e}^{2t}\sin(t), \quad c_1, c_2, c_3, c_4 \in \mathbb{R}. \qquad \boxed{1}$$

Remarks: a) was solved by most students. Some students produced a wrong factorization of $a(X)$, which didn't automatically result in the loss of all marks. (A fundamental system of solutions that is correct relative to the wrong factorization was normally honored by at least some marks.) But it usually lead to a false Ansatz in b), costing marks there. (Unsolvability of an equation in b) should ring a bell that something is wrong with the factorization in a).)

When grading b), it was very visible that students who knew about complexification and the computational trick to evaluate $a(\mathrm{D})y$ using the factorization of $a(X)$ (see the computation above) made less errors in the computations than those who computed the first 4 derivatives of the candidate functions $y_1(t)$, $y_2(t)$ and substituted these into the ODE's. In order to use the trick for the computation of $y_2(t)$, the corresponding ODE needs to be complexified first. You should have known about these things, because they were mentioned in the remarks accompanying the solutions to the first sample exam.

$$\sum_{6} = 8$$

$$\sum_{\text{Final Exam}} = 12 + 10 + 7 + 9 + 6 + 8 = 52 = 45 + 7$$