Question 1 (ca. 12 marks)

Decide whether the following statements are true or false, and justify your answers.

a) There exists a solution $y(t)$ of $y' = \ln \frac{y^2+1}{2}$ satisfying $y(0) = 0$, $y(3) = 3$.

b) The maximal solution $y(t)$ of the initial value problem $y' = y^2 + t$, $y(0) = 1$ is defined at $t = \frac{1}{2}$.

c) The ODE $(x^4 - 1)y'' + (x^2 - 1)y' + (x - 1)y = 0$ has a nonzero power series solution $y(x) = \sum_{n=0}^{\infty} a_n (x + 2)^n$ which is defined at $x = -4$.

d) Every solution of the system $\mathbf{y}' = \begin{pmatrix} -1 & -3 \\ 3 & 1 \end{pmatrix} \mathbf{y}$ satisfies $\lim_{t\to+\infty} \mathbf{y}(t) = (0, 0)^{\mathsf{T}}$.

e) If $\mathbf{A} \in \mathbb{R}^{3\times 3}$ satisfies $\mathbf{A}^3 = \mathbf{A}$ then $\mathrm{e}^{\mathbf{A}t} = \mathbf{I} + \sinh(t)\mathbf{A} + (\cosh t - 1)\mathbf{A}^2$. ($\mathbf{I}$ denotes the 3×3 identity matrix.)

f) Suppose $f, g \colon (0, \infty) \to \mathbb{R}$ are C^1-functions. Then the initial value problem $y' = f(t)g(y)$, $y(1) = 1$ has a solution $y(t)$ that is defined for all $t > 0$.

Question 2 (ca. 9 marks)

Consider the differential equation

$$2x^2y'' + (x^2 - 3x)y' + 2y = 0. \qquad \text{(DE)}$$

a) Verify that $x_0 = 0$ is a regular singular point of (DE).

b) Determine the general solution of (DE) on $(0, \infty)$.

c) Using the result of b), state the general solution of (DE) on $(-\infty, 0)$ and on $\mathbb{R}$.

Question 3 (ca. 6 marks)

For the initial value problem

$$y' = \frac{y + t}{2y - t}, \qquad y(2) = 2, \qquad \text{(H)}$$

determine the maximal solution $y(t)$ and its domain.

Hint: The substitution $z(t) = y(t)/t$ transforms (H) into a separable ODE. In order to see this, rewrite y' in terms of z. When solving the separable ODE, the formula $\int \frac{2az + b}{az^2 + bz + c}\,\mathrm{d}z = \ln\left|az^2 + bz + c\right| + C$ may be helpful.

Question 4 (ca. 8 marks)

Consider $\mathbf{A} = \begin{pmatrix} 3 & -2 & -1 \\ -1 & 1 & 0 \\ 8 & -6 & -2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.

a) Determine a fundamental system of solutions of the system $\mathbf{y}' = \mathbf{A}\mathbf{y}$.

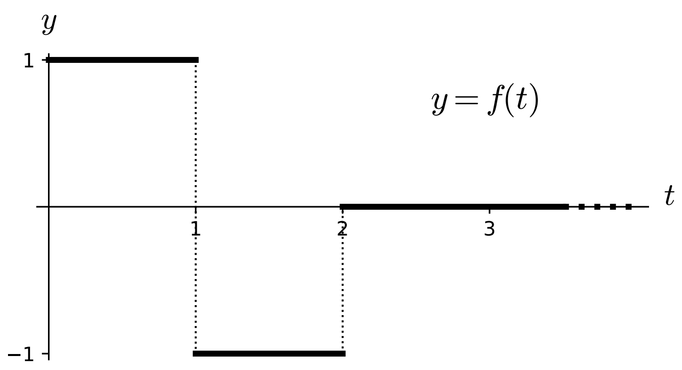
b) Solve the initial value problem $\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{b}$, $\mathbf{y}(0) = (0,0,0)^{\mathsf{T}}$.
Hint: There is a particular solution of the form $\mathbf{y}(t) = \mathbf{w}_0 + t\,\mathbf{w}_1$ ($\mathbf{w}_0, \mathbf{w}_1 \in \mathbb{R}^3$).

Question 5 (ca. 6 marks)

For the function f sketched below, solve the initial value problem

$$y'' + 2\,y' + y = f(t), \quad y(0) = 1,\ y'(0) = 0$$

with the Laplace transform.



Note: For the solution $y(t)$ explicit formulas valid in the intervals $[0,1]$, $[1,2]$, $[2,\infty)$ are required. You *must* use the Laplace transform for the computation.

Question 6 (ca. 6 marks)

a) Determine a real fundamental system of solutions of

$$y''' + y'' - 2\,y = 0.$$

b) Determine the general real solution of

$$y''' + y'' - 2\,y = 1 - 2\,t^3 + \mathrm{e}^{-t}\cos t.$$

Solutions

1 a) False: $y' = \ln\frac{y^2+1}{2}$ has the constant solution $y_1(t) \equiv 1$. Because of continuity, a solution $y_2(t)$ with the indicated property would have to attain the value 1. If $y_2(t_0) = 1$ then, on the domain of $y_2(t)$, we would have two distinct solutions of the IVP $y' = \ln\frac{y^2+1}{2}$, $y(t_0) = 1$, which according to the Existence and Uniqueness Theorem is impossible. [2]

b) True. Denoting the maximal domain by (a, b), we have $y'(t) > 0$ for $t \in [0, b)$, i.e., $y(t)$ is increasing on $[0, b)$. Thus, if b is finite, we must have $\lim_{t\uparrow b} y(t) = +\infty$. On the other hand, as long as $0 \leq t \leq 1$ and $y(t)$ exists, it is bounded from above by the solution $z(t)$ of $z' = z^2 + 1$, $z(0) = 1$, which is $z(t) = \tan(t + \pi/4)$ and exists for $t \in [0, \pi/4)$. Hence $b \geq \pi/4 > 1/2$, and $y(1/2)$ is well-defined. [2]

c) False. The point $x_0 = -2$ is an ordinary point, so that nonzero power series solutions $y(x)$ of the indicated form exist, but their guaranteed radius of convergence (and in fact the true radius of convergence) is only the distance from -2 to the nearest singularity of $q(x) = \frac{x-1}{x^4-1} = \frac{1}{(x+1)(x^2+1)}$, which is -1. Thus $R = 1$ and $y(x)$ is not defined at $x = -4$. [2]

d) False. As derived in the lecture, this is true iff the system is asymptotically stable, which in turn is the case iff the eigenvalues of $\mathbf{A} = \begin{pmatrix} -1 & -3 \\ 3 & 1 \end{pmatrix}$ have negative real part. But $\lambda_1 + \lambda_2 = \operatorname{tr}(\mathbf{A}) = -1 + 1 = 0$, contradiction! (In fact $\chi_{\mathbf{A}}(X) = X^2 + 8$, and $\lambda_{1/2} = \pm 2\sqrt{2}\,\mathrm{i}$ are purely imaginary. [2]

e) True. We have $a(\mathbf{A}) = \mathbf{0}$ for $a(X) = X^3 - X = X(X-1)(X+1)$. The ODE $a(\mathrm{D})y = 0$ has the fundamental system 1, e^t, e^{-t}, Hence 1, $\sinh t = \frac{1}{2}\mathrm{e}^t - \frac{1}{2}\mathrm{e}^{-t}$, $\cosh t - 1 = -1 + \frac{1}{2}\mathrm{e}^t + \frac{1}{2}\mathrm{e}^{-t}$ solve the ODE. Since the corresponding Wronski matrix is the 3×3 identity matrix, $\mathrm{e}^{\mathbf{A}t}$ admits the indicated representation; cf. lecture. [2]

f) False. Here is a counterexample: Take $f(t) = 1$, $g(y) = y^2$, so that the ODE is $y' = y^2$. Its solutions are $y(t) = 1/(C - t)$, $C \in \mathbb{R}$. The (maximal) solution satisfying $y(1) = 1$ is the one with $C = 2$, and is defined on $(-\infty, 2)$. Hence no solution of the IVP is defined at $t = 2$ (or at larger t). [2]

$\sum_1 = 12$

---

2 a) The explicit form of (DE) is

$$y'' + \left(\frac{1}{2} - \frac{3}{2x}\right)y' + \frac{1}{x^2}\,y = 0$$

$p(x) := \frac{1}{2} - \frac{3}{2x}$ has a pole of order 1 at 0, and $q(x) := \frac{1}{x^2}$ has a pole of order 2 at 0. This shows that 0 is a regular singular point of (DE). [1]

Alternatively, use that the limits defining p_0, q_0 below are finite.

b) From a) we have $p_0 = \lim_{x\to 0} x\,p(x) = -3/2$, $q_0 = \lim_{x\to 0} x^2\,q(x) = 1$. (These coefficients can just be read off from the explicit form.)
$\Longrightarrow$ The indicial equation is

$$r^2 + (p_0 - 1)r + q_0 = r^2 - \frac{5}{2}r + 1 = (r-2)(r-1/2) = 0.$$

$\Longrightarrow$ The exponents at the singularity $x_0 = 0$ are $r_1 = 2$, $r_2 = 1/2$. Since $r_1 - r_2 \notin \mathbb{Z}$, there exist two fundamental solutions y_1, y_2 of the form

$$\begin{aligned} y_1(x) &= x^2 \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+2}, \\ y_2(x) &= x^{1/2} \sum_{n=0}^{\infty} b_n x^n = \sum_{n=0}^{\infty} b_n x^{n+1/2} \end{aligned} \qquad \boxed{1}$$

with normalization $a_0 = b_0 = 1$.

First we determine $y_1(x)$. We have

$$\begin{aligned} 0 &= 2x^2 y_1'' + (x^2 - 3x)y_1' + 2\,y_1 \\ &= 2x^2 \sum_{n=0}^{\infty} (n+2)(n+1)a_n x^n + (x^2-3x)\sum_{n=0}^{\infty}(n+2)a_n x^{n+1} + 2\sum_{n=0}^{\infty} a_n x^{n+2} \\ &= \sum_{n=0}^{\infty} [2(n+2)(n+1) - 3(n+2) + 2]\, a_n x^{n+2} + \sum_{n=0}^{\infty}(n+2)a_n x^{n+3} \\ &= \sum_{n=0}^{\infty}(2n^2+3n)a_n x^{n+2} + \sum_{n=1}^{\infty}(n+1)a_{n-1}x^{n+2} \\ &= \sum_{n=1}^{\infty}[n(2n+3)a_n + (n+1)a_{n-1}]\, x^{n+2}. \end{aligned}$$

Equating coefficients gives the recurrence relation

$$a_n = -\frac{n+1}{n(2n+3)} a_{n-1} \quad \text{for } n = 1, 2, 3, \dots, \qquad \boxed{1}$$

and with $a_0 = 1$ further $a_n = (-1)^n \frac{n+1}{5\cdot 7\cdot 9\cdots(2n+3)}$ for $n \geq 1$.

$$\begin{aligned} \Longrightarrow y_1(x) &= \sum_{n=0}^{\infty} (-1)^n \frac{n+1}{5\cdot 7\cdot 9\cdots(2n+3)}\, x^{n+2} \qquad \boxed{1\tfrac{1}{2}} \\ &= x^2 - \frac{2}{5}x^3 + \frac{3}{5\cdot 7}x^4 - \frac{4}{5\cdot 7\cdot 9}x^5 + \frac{5}{5\cdot 7\cdot 9\cdot 11}x^6 \mp \cdots \end{aligned}$$

(For $n = 1$ the product in the denominator is understood as the the empty product 1.)

For the determination of $y_2(x)$ we repeat the process with exponents decrreased by

1.5:

$$\begin{aligned}
0 &= 2x^2\,y_2'' + (x^2-3x)y_2' + 2\,y_2\\
&= 2x^2\sum_{n=0}^{\infty}\left(n+\tfrac{1}{2}\right)\left(n-\tfrac{1}{2}\right)b_n x^{n-3/2} + (x^2-3x)\sum_{n=0}^{\infty}\left(n+\tfrac{1}{2}\right)b_n x^{n-1/2} + 2\sum_{n=0}^{\infty}b_n x^{n+1/2}\\
&= \sum_{n=0}^{\infty}\left[2\left(n+\tfrac{1}{2}\right)\left(n-\tfrac{1}{2}\right) - 3\left(n+\tfrac{1}{2}\right)+2\right]b_n x^{n+1/2} + \sum_{n=0}^{\infty}\left(n+\tfrac{1}{2}\right)b_n x^{n+3/2}\\
&= \sum_{n=0}^{\infty}(2n^2-3n)b_n x^{n+1/2} + \sum_{n=1}^{\infty}\left(n-\tfrac{1}{2}\right)b_{n-1}x^{n+1/2}\\
&= \sum_{n=1}^{\infty}\left[n(2n-3)b_n + \left(n-\tfrac{1}{2}\right)b_{n-1}\right]x^{n+1/2}.
\end{aligned}$$

Here we obtain the recurrence relation

$$b_n = -\frac{n-\frac{1}{2}}{n(2n-3)}\,b_{n-1} = -\frac{2n-1}{2n(2n-3)}\,b_{n-1} \quad \text{for } n=1,2,3,\dots, \qquad \boxed{1}$$

and with $b_0=1$ further $b_n = (-1)^n\frac{1\cdot3\cdot5\cdots(2n-1)}{2\cdot4\cdot6\cdots(2n)(-1)1\cdot3\cdots(2n-3)} = (-1)^{n-1}\frac{2n-1}{2\cdot4\cdot6\cdots(2n)}$ for $n\geq1$.

$$\begin{aligned}
\implies\ y_2(x) &= x^{1/2} + \sum_{n=1}^{\infty}(-1)^{n-1}\frac{2n-1}{2\cdot4\cdot6\cdots(2n)}\,x^{n+1/2} \qquad \boxed{1\tfrac{1}{2}}\\
&= x^{1/2} + \frac{1}{2}\,x^{3/2} - \frac{3}{2\cdot4}\,x^{5/2} + \frac{5}{2\cdot4\cdot6}\,x^{7/2} - \frac{7}{2\cdot4\cdot6\cdot8}\,x^{9/2} \mp \cdots
\end{aligned}$$

Alternative solution: We use the general recurrence relation for the rational functions $a_n(r)$, viz. $a_0(r)=1$ and

$$a_n(r) = -\frac{1}{F(r+n)}\sum_{k=0}^{n-1}\left[(r+k)p_{n-k}+q_{n-k}\right]a_k(r) \quad \text{for } n\geq1.$$

Since $F(r)=(r-2)(r-1/2)$ and all coefficients p_i,q_i except for p_0,q_0 and $p_1=1/2$ are zero, we obtain

$$\begin{aligned}
a_n(r) &= -\frac{(r+n-1)p_1}{(r+n-2)(r+n-1/2)}\,a_{n-1}(r)\\
&= -\frac{r+n-1}{(r+n-2)(2r+2n-1)} \quad \text{for } n\geq1.
\end{aligned}$$

Thus the coefficients $a_n(2)$ of $y_1(x)$ satisfy the recurrence relation $a_n(2) = -\frac{n+1}{n(2n+3)}\,a_{n-1}(2)$ (the same as for a_n above) and the coefficients $a_n(1/2)$ of $y_2(x)$ satisfy the recurrence relation $a_n(1/2) = -\frac{n-1/2}{(n-3/2)2n}\,a_{n-1}(1/2) = -\frac{2n-1}{(2n-3)2n}\,a_{n-1}(1/2)$ (the same as for b_n above). The rest of the computation remains the same.

The general (real) solution on $(0,\infty)$ is then $y(x)=c_1\,y_1(x)+c_2\,y_2(x)$, $c_1,c_2\in\mathbb{R}$. $\boxed{\tfrac{1}{2}}$

That solutions are defined on the whole of $(0,\infty)$, is guaranteed by the analyticity of $p(x)$, $q(x)$ in $\mathbb{C}\setminus\{0\}$, but follows also readily from the easily established fact that the radius of convergence of both power series is ∞. $\boxed{\tfrac{1}{2}}$

c) The solution on $(-\infty,0)$ is $y(x)=c_1\,y_1(x)+c_2\,y_2^-(x)$ with the same power series $y_1(x)$ as in b) and

$$y_2^-(x)=(-x)^{1/2}\sum_{n=0}^{\infty}\frac{(-1)^{n-1}(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}\,x^n. \qquad \boxed{1}$$

(This is <u>not</u> the same as $y_2(-x)$, which has negative coefficients when written in terms of powers of $-x$.)

Since $y_1(x)$ is analytic at zero but $y_2(x)$ is not, the general solution on $\mathbb{R}$ is $y(x)=c\,y_1(x)$, $c\in\mathbb{R}$. $\boxed{1}$

$$\sum_2 = 10$$

3 Suppressing the argument t as usual, we have $y'=\frac{y/t+1}{2y/t-1}=\frac{z+1}{2z-1}$ and hence

$$z'=\left(\frac{y}{t}\right)'=\frac{y't-y}{t^2}=\frac{y'-z}{t}=\frac{1}{t}\left(\frac{z+1}{2z-1}-z\right)=\frac{-2z^2+2z+1}{t(2z-1)}. \qquad \boxed{2}$$

This is a separable equation and can be solved by the usual method, noting that $y(2)=2$ corresponds to $z(2)=1$:

$$\begin{aligned}
\frac{2z-1}{-2z^2+2z+1}\,\mathrm{d}z &= \frac{\mathrm{d}t}{t}\\
\int_1^z\frac{2\zeta-1}{-2\zeta^2+2\zeta+1}\,\mathrm{d}\zeta &= \int_2^t\frac{\mathrm{d}\tau}{\tau}\\
\left[-\frac{1}{2}\ln\left|-2\zeta^2+2\zeta+1\right|\right]_1^z &= \left[\ln|\tau|\right]_2^t\\
-\frac{1}{2}\ln\left(-2z^2+2z+1\right) &= \ln t-\ln 2=\ln\frac{t}{2}\\
\ln\left(-2z^2+2z+1\right) &= -2\ln\frac{t}{2}\\
-2z^2+2z+1 &= \mathrm{e}^{-2\ln\frac{t}{2}}=\frac{4}{t^2}\\
2z^2-2z-1+\frac{4}{t^2} &= 0\\
z &= \frac{1}{4}\left(2\pm\sqrt{4-8\left(\frac{4}{t^2}-1\right)}\right)=\frac{1}{2}\left(1\pm\sqrt{3-\frac{8}{t^2}}\right) \qquad \boxed{3}
\end{aligned}$$

Since $z(2)=1$, the correct sign is '+'. The solution of (H) is then

$$y(t)=t\,z(t)=\frac{t}{2}\left(1+\sqrt{3-\frac{8}{t^2}}\right) \qquad \boxed{1}$$

with maximal domain determined by $3-8/t^2>0$, i.e., $t>\sqrt{8/3}=\frac{2\sqrt{2}}{\sqrt{3}}$ (since it must be an interval containing $t=2$). $\boxed{1}$.

$$\sum_3 = 7$$

4 a) The characteristic polynomial of $\mathbf{A}$ is

$$\chi_{\mathbf{A}}(X)=\begin{vmatrix} X-3 & 2 & 1\\ 1 & X-1 & 0\\ -8 & 6 & X+2\end{vmatrix}=\begin{vmatrix} X-3 & 2 & 1\\ 1 & X-1 & 0\\ -2+X-X^2 & 2-2X & 0\end{vmatrix}$$

$$=\begin{vmatrix} X-3 & 2 & 1\\ 1 & X-1 & 0\\ X-X^2 & 0 & 0\end{vmatrix}=(-1)(X-X^2)(X-1)=X(X-1)^2.$$

$\Longrightarrow$ The eigenvalues of $\mathbf{A}$ are $\lambda_1=0$ with algebraic multiplicity 1 and $\lambda_2=1$ with algebraic multiplicity 2. $\boxed{2}$

$$\mathbf{A}-0\,\mathbf{I}=\begin{pmatrix} 3 & -2 & -1\\ -1 & 1 & 0\\ 8 & -6 & -2\end{pmatrix}\to\begin{pmatrix} -3 & 2 & 1\\ 1 & -1 & 0\\ 0 & 0 & 0\end{pmatrix}$$

(Substitute $X=0$ in the computation above.)
$\Longrightarrow$ The eigenspace corresponding to $\lambda_2=1$ is one-dimensional and generated by $\mathbf{v}_2=(1,1,1)^\mathsf{T}$. (This is also clear from the fact that $\mathbf{A}$ has constant row sums zero.)

$$\mathbf{A}-\mathbf{I}=\begin{pmatrix} 2 & -2 & -1\\ -1 & 0 & 0\\ 8 & -6 & -3\end{pmatrix}\to\begin{pmatrix} -2 & 2 & 1\\ 1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$$

(Substitute $X=1$ in the computation above.)
$\Longrightarrow$ The eigenspace corresponding to $\lambda_1=1$ is one-dimensional and generated by $\mathbf{v}_2=(0,1,-2)^\mathsf{T}$.

A further generalized eigenvector $\mathbf{v}_3$ can be found by solving $(\mathbf{A}-\mathbf{I})\mathbf{v}_3=\mathbf{v}_2$:

$$\left(\begin{array}{ccc|c} 2 & -2 & -1 & 0\\ -1 & 0 & 0 & 1\\ 8 & -6 & -3 & -2\end{array}\right)\to\left(\begin{array}{ccc|c} -1 & 0 & 0 & 1\\ 0 & -2 & -1 & 2\\ 0 & -6 & -3 & 6\end{array}\right)\to\left(\begin{array}{ccc|c} -1 & 0 & 0 & 1\\ 0 & -2 & -1 & 2\\ 0 & 0 & 0 & 0\end{array}\right),$$

e.g., $\mathbf{v}_3=(-1,-1,0)^\mathsf{T}$.

The corresponding fundamental system of solutions is:

$$\mathbf{y}_1(t)=\begin{pmatrix}1\\1\\1\end{pmatrix},$$

$$\mathbf{y}_2(t)=\mathrm{e}^t\begin{pmatrix}0\\1\\-2\end{pmatrix},$$

$$\mathbf{y}_3(t)=\mathrm{e}^t\begin{pmatrix}-1\\-1\\0\end{pmatrix}+t\,\mathrm{e}^t\begin{pmatrix}0\\1\\-2\end{pmatrix}.$$ $\boxed{3}$

Changing signs in $\mathbf{y}_3(t)$, i.e., choosing $(0,-1,2)$ as generator of the eigenspace for $\lambda_2=1$ makes the figures slightly simpler.

b) $\mathbf{y}(t) = \mathbf{w}_0 + t\,\mathbf{w}_1$ is a solution iff $\mathbf{w}_1 = \mathbf{y}'(t) = \mathbf{A}\mathbf{y}(t) + \mathbf{b} = \mathbf{A}\mathbf{w}_0 + t\,\mathbf{A}\mathbf{w}_1 + \mathbf{b}$, which is equivalent to $\mathbf{w}_1 = \mathbf{A}\mathbf{w}_0 + \mathbf{b} \wedge \mathbf{A}\mathbf{w}_1 = \mathbf{0}$. Thus we need to solve $\mathbf{A}^2\mathbf{w}_0 + \mathbf{A}\mathbf{b} = \mathbf{0}$. $\boxed{1}$

$$\mathbf{A}^2 = \begin{pmatrix} 3 & -2 & -1 \\ -4 & 3 & 1 \\ 14 & -10 & -4 \end{pmatrix}, \quad \mathbf{A}\mathbf{b} = \begin{pmatrix} -2 \\ 1 \\ -6 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 3 & -2 & -1 & 2 \\ -4 & 3 & 1 & -1 \\ 14 & -10 & -4 & 6 \end{array}\right) \to \left(\begin{array}{ccc|c} -1 & 1 & 0 & 1 \\ -4 & 3 & 1 & -1 \\ -2 & 2 & 0 & 2 \end{array}\right) \to \left(\begin{array}{ccc|c} -1 & 1 & 0 & 1 \\ -4 & 3 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

The solution with $x_1 = 0$ is $x_2 = 1$, $x_3 = -4$, i.e., $\mathbf{w}_0 = (0, 1, -4)^\mathsf{T}$, giving

$$\mathbf{w}_1 = \begin{pmatrix} 3 & -2 & -1 \\ -1 & 1 & 0 \\ 8 & -6 & -2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ -4 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix},$$

$$\mathbf{y}_p(t) = \begin{pmatrix} 0 \\ 1 \\ -4 \end{pmatrix} + t \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}. \qquad \boxed{2}$$

The general solution of $\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{b}$ is $\mathbf{y}(t) = \mathbf{y}_p(t) + c_1\mathbf{y}_1(t) + c_2\mathbf{y}_2(t) + c_3\mathbf{y}_3(t)$. In order to satisfy the required initial condition, (c_1, c_2, c_3) needs to solve

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 1 & 1 & -1 & -1 \\ 1 & -2 & 0 & 4 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & -2 & 1 & 4 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

$\Longrightarrow c_3 = 2$, $c_2 = -1$, $c_1 = 2$, and the final answer is

$$\begin{aligned} y(t) &= 2\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \mathrm{e}^t \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix} + 2\,\mathrm{e}^t \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} + 2t\,\mathrm{e}^t \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ -4 \end{pmatrix} + t \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} \\ &= \begin{pmatrix} 2 + 2t - 2\,\mathrm{e}^t \\ 3 + 2t - 3\,\mathrm{e}^t + 2t\,\mathrm{e}^t \\ -2 + 2t + 2\,\mathrm{e}^t - 4t\,\mathrm{e}^t \end{pmatrix}. \end{aligned} \qquad \boxed{2}$$

$$\sum_4 = 10$$

---

5 Writing $Y(s) = \mathcal{L}\{y(t)\}$, $F(s) = \mathcal{L}\{f(t)\}$, and applying the Laplace transform to both sides of the ODE gives

$$\begin{aligned} \mathcal{L}\{y'' + 2\,y' + y\} &= s^2\,Y(s) - s\,y(0) + 2\big(s\,Y(s) - y(0)\big) + Y(s) \\ &= (s^2 + 2s + 1)Y(s) - s - 2 = \mathcal{L}\big\{f(t)\big\} = F(s). \end{aligned}$$

Further we have

$$\begin{aligned} f(t) &= \mathrm{u}(t) - \mathrm{u}(t-1) - \big(\mathrm{u}(t-1) - \mathrm{u}(t-2)\big) \\ &= \mathrm{u}(t) - 2\mathrm{u}(t-1) + \mathrm{u}(t-2), & \boxed{1} \\ \implies F(s) &= \frac{1 - 2\,\mathrm{e}^{-s} + \mathrm{e}^{-2s}}{s}. & \boxed{1} \\ \implies Y(s) &= \frac{s+2}{(s+1)^2} + \frac{1 - 2\,\mathrm{e}^{-s} + \mathrm{e}^{-2s}}{s(s+1)^2} & \boxed{1} \\ &= \frac{1}{s} + \frac{-2\,\mathrm{e}^{-s} + \mathrm{e}^{-2s}}{s(s+1)^2} \\ &= \frac{1}{s} + \left(\frac{1}{s} - \frac{1}{s+1} - \frac{1}{(s+1)^2}\right)(-2\,\mathrm{e}^{-s} + \mathrm{e}^{-2s}) & \boxed{1} \end{aligned}$$

The inverse Laplace transform of $\frac{1}{s} - \frac{1}{s+1} - \frac{1}{(s+1)^2}$ is $1 - \mathrm{e}^{-t} - t\,\mathrm{e}^{-t}$.

$$\begin{aligned} \implies y(t) &= 1 - 2\,\mathrm{u}_1(t)\left(1 - \mathrm{e}^{1-t} + (1-t)\mathrm{e}^{1-t}\right) + \mathrm{u}_2(t)\left(1 - \mathrm{e}^{2-t} + (2-t)\mathrm{e}^{2-t}\right) \\ &= 1 - 2\,\mathrm{u}_1(t)\left(1 - t\,\mathrm{e}^{1-t}\right) + \mathrm{u}_2(t)\left(1 + \mathrm{e}^{2-t} - t\,\mathrm{e}^{2-t}\right) & \boxed{1} \\ &= \begin{cases} 1 & \text{for } 0 \le t \le 1, \\ -1 + 2t\,\mathrm{e}^{1-t} & \text{for } 1 \le t \le 2, \\ 2t\,\mathrm{e}^{1-t} + \mathrm{e}^{2-t} - t\,\mathrm{e}^{2-t} & \text{for } t \ge 2. \end{cases} & \boxed{1} \end{aligned}$$

The 3rd expression can also be written as $(2t + \mathrm{e} - t\,\mathrm{e})\mathrm{e}^{1-t}$.

$$\sum_5 = 6$$

---

6 a) The characteristic polynomial is

$$\begin{aligned} a(X) &= X^3 + X^2 - 2 \\ &= (X-1)(X^2 + 2X + 2) \\ &= (X-1)(X+1-\mathrm{i})(X+1+\mathrm{i}). \end{aligned}$$

with zeros $\lambda_1 = 1$, $\lambda_2 = -1+\mathrm{i}$, $\lambda_3 = -1-\mathrm{i}$, all of multiplicity 1. $\boxed{1}$

$\implies$ A complex fundamental system of solutions is e^t, $\mathrm{e}^{(-1+\mathrm{i})t}$, $\mathrm{e}^{(-1-\mathrm{i})t}$, and the corresponding real fundamental system is

$$\mathrm{e}^t, \quad \mathrm{e}^{-t}\cos t, \quad \mathrm{e}^{-t}\sin t. \qquad \boxed{1\tfrac{1}{2}}$$

b) In order to obtain a particular solution $y_p(t)$ of the inhomogeneous equation, we solve the two equations $a(\mathrm{D})y_i = b_i(t)$ for $b_1(t) = 1 - 2\,t^3$, $b_2(t) = \mathrm{e}^{-t}\mathrm{e}^{\mathrm{i}t} = \mathrm{e}^{(-1+\mathrm{i})t}$. Superposition then yields the particular solution $y_p(t) = y_1(t) + \operatorname{Re} y_2(t)$.

(1) Since $\mu = 0$ is not a root of $a(X)$, the correct Ansatz is $y_1(t) = c_0 + c_1 t + c_2 t^2 + c_3 t^3$.

$$\begin{aligned} y_1''' + y_1'' - 2\,y_1 &= 6c_3 + 2c_2 + 6c_3 t - 2(c_0 + c_1 t + c_2 t^2 + c_3 t^3) \\ &= 6c_3 + 2c_2 - 2c_0 + (6c_3 - 2c_1)t - 2c_2 t^2 - 2c_3 t^3 \stackrel{!}{=} 1 - 2\,t^3 \end{aligned}$$

$\implies c_3 = 1$, $c_2 = 0$, $c_1 = 3c_3 = 3$, $c_0 = (6c_3 + 2c_2 - 1)/2 = 5/2$, so that $y_1(t) = \frac{5}{2} + 3\,t + t^3$. $\boxed{1}$

(2) Since $\mu = -1 + \mathrm{i}$ is a zero of $a(X)$ of multiplicity 1, the correct Ansatz is $y_2(t) = ct\,\mathrm{e}^{(-1+\mathrm{i})t}$.

$$\begin{aligned} y_2''' + y_2'' - 2\,y_2 &= (\mathrm{D}-1)(\mathrm{D}+1+\mathrm{i})(\mathrm{D}+1-\mathrm{i})\left[ct\,\mathrm{e}^{(-1+\mathrm{i})t}\right] \\ &= c(\mathrm{D}-1)(\mathrm{D}+1+\mathrm{i})\mathrm{e}^{(-1+\mathrm{i})t} \\ &= c(\mathrm{D}-1)\left[2\mathrm{i}\,\mathrm{e}^{(-1+\mathrm{i})t}\right] \\ &= c\,2\mathrm{i}(-2+\mathrm{i})\mathrm{e}^{(-1+\mathrm{i})t} = c(-2-4\mathrm{i})\mathrm{e}^{(-1+\mathrm{i})t} \end{aligned}$$

$\Longrightarrow c = \frac{1}{-2-4\mathrm{i}} = \frac{-2+4\mathrm{i}}{2^2+4^2} = \frac{-1+2\mathrm{i}}{10} \Longrightarrow y_2(t) = \frac{-1+2\mathrm{i}}{10}\,t\,\mathrm{e}^{(-1+\mathrm{i})t}$. $\boxed{1\frac{1}{2}}$

Putting things together gives

$$\begin{aligned} y_p(t) &= \frac{5}{2} + 3\,t + t^3 + \mathrm{Re}\left(\frac{-1+2\mathrm{i}}{10}\,t\,\mathrm{e}^{(-1+\mathrm{i})t}\right) \\ &= \frac{5}{2} + 3\,t + t^3 - \frac{1}{10}\,t\,\mathrm{e}^{-t}\cos t - \frac{1}{5}\,t\,\mathrm{e}^{-t}\sin t. \end{aligned} \qquad \boxed{1}$$

The general real solution is then

$$y(t) = y_p(t) + c_1\mathrm{e}^t + c_2\mathrm{e}^{-t}\cos t + c_3\mathrm{e}^{-t}\sin t, \quad c_1, c_2, c_3 \in \mathbb{R}. \qquad \boxed{1}$$

$$\sum_{6} = 7$$

$$\sum_{\text{Final Exam}} = 52$$
