question	1	2	3	4	5	Total
score						

MATH213 Second Midterm
Fall 2023

NAME: ______________________

Instructor: M. Zhang

Please answer all first *four* questions (question 5 is optional).

- You are allowed one double-sided cheat sheet.
- Show all work for full credit.
- No calculators are permitted.
- Please write your name on all the page.

Good luck!

1. (30 points) Let F_n be the Fibonacci number, i.e., $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$.

   (a) 15 pts Prove that $F_0F_1 + F_1F_2 + \cdots + F_{2n-1}F_{2n} = F_{2n}^2$, where n is a positive integer.

   (b) 15 pts Prove by mathematical induction that

$$\sum_{k=0}^{n} \binom{n-k}{k} = F_{n+1}$$

Solutions

(a) 15 pts **Basis**: $f_0 f_1 + f_1 f_2 = 0 \cdot 1 + 1 \cdot 1 = 1^2 = f_2^2$.

Induction: Assume that $f_0 f_1 + f_1 f_2 + \cdots + f_{2m-1} f_{2m} = f_{2m}^2$ holds for all $m \le k$. Then $f_0 f_1 + f_1 f_2 + \cdots + f_{2k-1} f_{2k} + f_{2k+1} f_{2k+2} = f_{2k}^2 + f_{2k+1} f_{2k+2} = f_{2k}(f_{2k} + f_{2k+1}) + f_{2k+1} f_{2k+2} = (f_{2k} + f_{2k+1}) f_{2k+2} = f_{2k+2}^2$.

(b) 15 pts

Basis: show that $n = 0$, $n = 1$, the equality holds.

Note that you have to explicitly show at least 2 bases, and it is because the Fibonacci is a second order recursion. If you show only one basis k, then how would $F_{k+1} = F_k + F_{k-1}$ hold since $k - 1$ is not examined?

Induction: Suppose that for all $m \le n$, the equality holds, and now prove the equality holds for $n + 1$.

$$
\begin{aligned}
F_{n+1} = F_n + F_{n-1} & \\
& = \sum_{k=0}^{n} \binom{n-k}{k} + \sum_{k=0}^{n-1} \binom{n-1-k}{k} \\
& = \sum_{k=0}^{n} \binom{n-k}{k} + \sum_{k=1}^{n} \binom{n-k}{k-1} \\
& = \sum_{k=0}^{n} \binom{n-k}{k} + \sum_{k=1}^{n} \binom{n-k}{k-1} \\
& = \sum_{k=0}^{n} \binom{n+1-k}{k} + \binom{n+1-(n+1)}{n+1} \\
& = \sum_{k=0}^{n+1} \binom{n+1-k}{k}
\end{aligned}
$$

Thus, by induction, the equality holds for all $n \ge 0$.

2. (30 points) Find all solutions of the recurrence relation $a_n = 4a_{n-1} - 4a_{n-2} + 3^n$. (Since we do not provide the initial conditions, obtain the general form would be sufficient.)

Solution

for $a_n^{(h)}$: $r^3 = 4r^2 - 4r$, we can get the root $r_1 = 0$,$r_2 = r_3 = 2$, therefore, $a_n^{(h)} = (\alpha_1 + \alpha_2 n)2^n$

for $a_n^{(p)}$: since 3 is not the root, $a_n^{(p)} = c3^n$. From $c3^n = 4c3^{n-1} - 4c3^{n-2} + 3^n$, we get $c = 9$, therefore, $a_n^{(p)} = 3^{n+2}$

Therefore, $a_n = a_n^{(h)} + a_n^{(p)} = (\alpha_1 + \alpha_2 n)2^n + 3^{n+2}$

Please notice that if you want to calculate the actual value for α_1 and α_2, you need to first calculate $a_n^{(p)}$, and then substitute it into a_0,a_1. In this case, $\begin{cases} a_0 & = \alpha_1 + 9 \\ a_1 & = 2(\alpha_1 + \alpha_2) + 27 \end{cases}$, we get $a_n = a_n^{(h)} + a_n^{(p)} = (a_0 - 9 + (\frac{a_1 - 27}{2} - a_0 + 9)n)2^n + 3^{n+2}$

3. (20 points) There are 25 girls and 25 boys seated around a circular table. Show that there is always a person X such that both of his/her neighbors are boys.

Sample Solution

Assume that the seats at the table are numbered consecutively from 1 to 50 in a clockwise direction. For the sake of argument, let's consider seat 50 to be adjacent to seat 1, completing the circle.

Notice that there are an equal number of girls and boys. Let's divide the seats into two categories based on their numbering: odd-numbered seats (1, 3, 5, ..., 49) and even-numbered seats (2, 4, 6, ..., 50).

By the Pigeonhole Principle, if we were to distribute the boys such that no more than 12 are seated in odd-numbered seats, then due to the equal number of boys and seats in each category, we must have at least 13 boys in the even-numbered seats. Conversely, if we place more than 12 boys in odd-numbered seats, there must be at least 13 boys there.

Without loss of generality, we assume at least 13 boys are seated in odd-numbered seats. Since there are only 25 odd-numbered seats and we have 13 boys, it's inevitable that at least two boys will be sitting in consecutive odd-numbered seats because there are not enough girls (only 12) to separate all boys.

Consider two boys seated at two such consecutive odd-numbered seats, say at seats $2k-1$ and $2k+1$ for $k \in \mathbb{Z}$ The person X seated at seat $2k$ (which is an even-numbered seat) will then have these two boys as neighbors.

Therefore, it is guaranteed that there will be at least one person X with boys seated on both sides, proving the statement.

This conclusion holds true regardless of the initial distribution of boys and girls around the table.

There are multiple valid solution for this question, you can ask regrade if you claim your answer has correct and clear logic.

4. (20 points)

Given set $B = \{a, b, c, d, e, f, g\}$, and relation R defined on B:

$$R = \{(a, c), (a, d), (b, f), (c, d), (d, d), (e, b), (e, f), (g, b), (g, f), (g, g)\}$$

$=$

(a) 4 pts Is relation R reflexive? Justify your answer.

(b) 4 pts Is relation R irreflexive? Justify your answer.

(c) 4 pts Is relation R symmetric? Justify your answer.

(d) 4 pts Is relation R antisymmetric? Justify your answer.

(e) 4 pts Is relation R transitive? Justify your answer.

Solution

(a) 4 pts False.

(b) 4 pts False.

(c) 4 pts False.

(d) 4 pts True.

(e) 4 pts True.

(a) Is R reflexive? - For R to be reflexive with resp;ect to set $B = \{a, b, c, d, e, f, g\}$, we need pairs like $(a, a), (b, b), (c, c), (d, d), (e, e), (f, f), (g, g)$ in R. - In R, only (d, d) and (g, g) satisfy this criterion. - Therefore, R is **not reflexive** because it does not include all necessary reflexive pairs.

(b) Is R irreflexive? - R contains the pairs (d, d) and (g, g), which are reflexive pairs. - Therefore, R is **not irreflexive** because it contains reflexive pairs.

(c) Is R symmetric? - R contains the pair (a, c), but not (c, a); (a, d), but not (d, a); (b, f), but not (f, b); and so on. - Therefore, R is **not symmetric** because it does not contain all corresponding symmetric pairs.

(d) Is R antisymmetric? - R contains (d, d) and (g, g), which satisfy the antisymmetric property since $x = y$. - No other pairs in R violate the antisymmetric property. - Therefore, R is **antisymmetric**.

(e) Is R transitive? - R contains (a, c) and (c, d), and also contains (a, d), satisfying transitivity for this case. - There are no other cases in R that violate the transitive property. - Therefore, R is **transitive**.

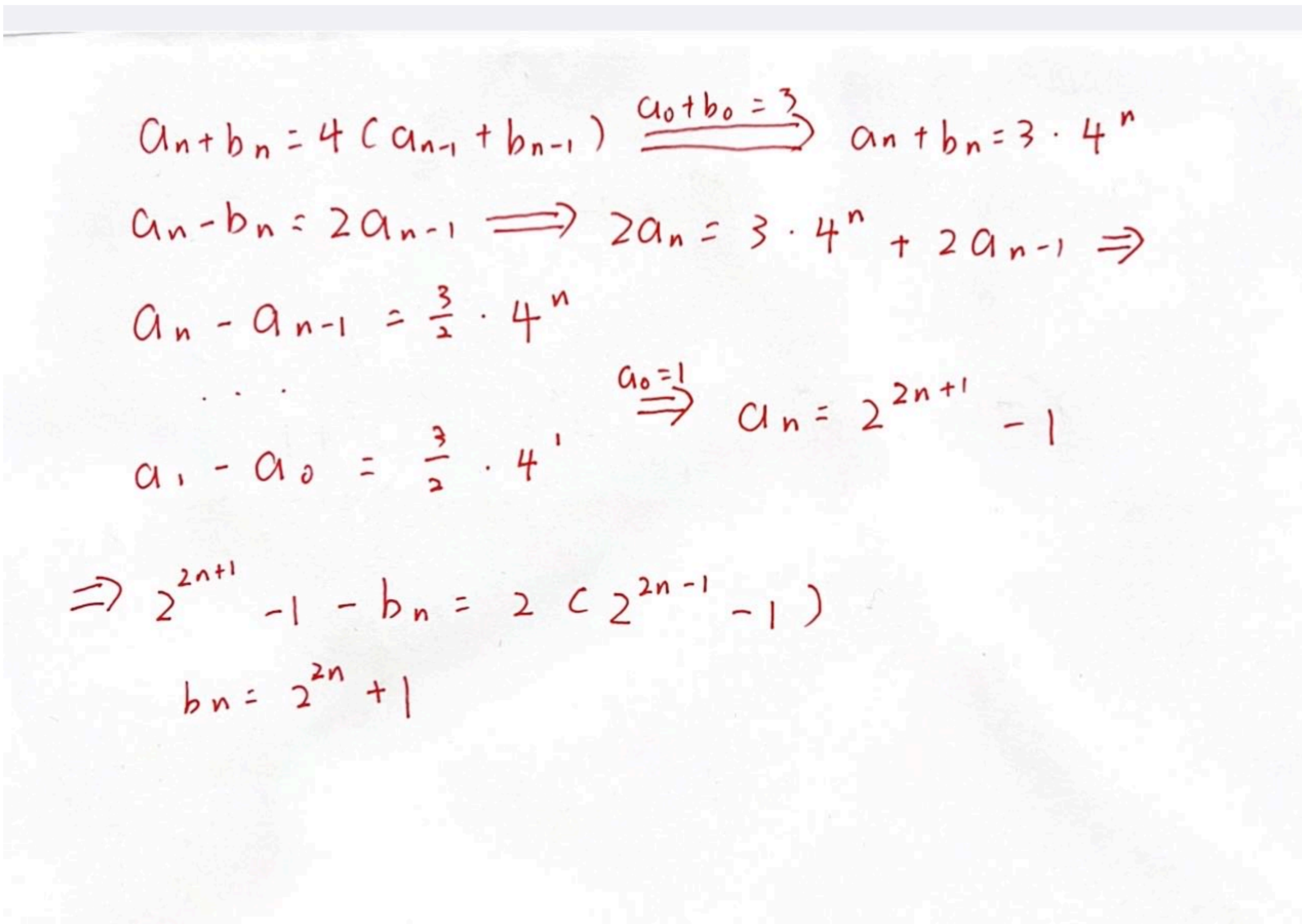
In conclusion: - (a) **False** - R is not reflexive. - (b) **False** - R is not irreflexive. - (c) **False** - R is not symmetric. - (d) **True** - R is antisymmetric. - (e) **True** - R is transitive.

2 points for judgement and 2 points for proof. You should combine You need to list all the cases for full points.

5. (Bonus 15 points) Solve the simultaneous recurrent relations:

$$\begin{aligned} a_n &= 3a_{n-1} + 2b_{n-1} \\ b_n &= a_{n-1} + 2b_{n-1} \end{aligned}$$

with $a_0 = 1$ and $b_0 = 2$.



$$a_n + b_n = 4(a_{n-1} + b_{n-1}) \xrightarrow{a_0 + b_0 = 3} a_n + b_n = 3 \cdot 4^n$$

$$a_n - b_n = 2a_{n-1} \Longrightarrow 2a_n = 3 \cdot 4^n + 2a_{n-1} \Rightarrow$$

$$a_n - a_{n-1} = \frac{3}{2} \cdot 4^n$$

$$\cdots \qquad \overset{a_0 = 1}{\Longrightarrow} \quad a_n = 2^{2n+1} - 1$$

$$a_1 - a_0 = \frac{3}{2} \cdot 4^1$$

$$\Rightarrow 2^{2n+1} - 1 - b_n = 2(2^{2n-1} - 1)$$

$$b_n = 2^{2n} + 1$$

Figure 1: solution