question	1	2	3	4	5	Total
score						

MATH213 Second Midterm
Fall 2022

NAME: ______________________

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Please answer all first *four* questions (question 5 is optional).

- You are allowed one double-sided cheat sheet.
- Show all work for full credit.
- Each is of equal worth (sub-problems within a problem are of equal worth).
- No calculators are permitted.

Good luck!

Answer:

1. Consider the statement: $C(n,k) \le 2^n$ holds for all $n \in \mathbb{N}$, $n \ge 1$ and all k with $0 \le k \le n$. (25 points)

   (a) Prove the statement based on the Pascal's identity;

   (b) Prove the statement based on the Binomial theorem.

   Solutions:

   (a) The Pascal's identity is

   $$\binom{n}{k+1} + \binom{n}{k} = \binom{n+1}{k+1},$$

   - Basic step: $C(1,0) = C(1,1) = 1 = 2^0$, which implies that the Pascal's identity is satisfied in this case.
   - Induction step: Assume that $C(n,k) \le 2^n$ for all first n times. We see that $\binom{n+1}{k+1} = \binom{n}{k+1} + \binom{n}{k} \le 2 \cdot 2^n = 2^{n+1}$.

   (b) From the Binomial theorem, we have $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$, which implies that

   $$2^n = (1+1)^n = \sum_{k=0}^{n} \binom{n}{k} \ge \binom{n}{k}.$$

- (20 points)
  - The **Lucas numbers** satisfy the recurrence relation:

$$L_n = L_{n-1} + L_{n-2},$$

and the initial conditions $L_0 = 2$ and $L_1 = 1$.

(a) Show that

$$L_n = f_{n-1} + f_{n+1}$$

for $n = 2, 3, ...$, where f_n is the nth **Fibonacci number** ($f_n = f_{n-1} + f_{n-2}$ for $n = 2, 3, ...$, with the initial conditions $f_0 = 0$ and $f_1 = 1$).

(b) Find an explicit formula for the Lucas numbers.

Solutions:

The characteristic function is $r^2 - r - 1 = 0$, with roots $r_1 = \frac{1-\sqrt{5}}{2}$ and $r_2 = \frac{1+\sqrt{5}}{2}$. So we must have

$$L_n = \alpha_1 r_1^n + \alpha_2 r_2^n,$$

and have $\alpha_1 = 1$ and $\alpha_2 = 2$.

$$L_n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n.$$

- (30 points)

  (a) How many ordered pairs of integers (a, b) are needed to guarantee that there are two ordered pairs (a_1, b_1) and (a_2, b_2) such that $a_1 \textbf{ mod } 5 = a_2 \textbf{ mod } 5$ and $b_1 \textbf{ mod } 5 = b_2 \textbf{ mod } 5$?

  (b) How many numbers must be selected from the set $\{1, 2, 3, 4, 5, 6\}$ to guarantee that at least one pair of these numbers add up to 7?

Solution:

(a) 26.
For each pair (a, b), there are five possible values of $a \textbf{ mod } 5$ and five possible values of $b \textbf{ mod } 5$ in total. Therefore, there are 25 possible distinct pairs. By the pigeonhole principle, we need 26 pairs.

(b) 4.
There are 3 pairs that add up to 7, i.e., $\{(1, 6), (5, 2), (4, 3)\}$. So the question is equivalent to how numbers must be selected from the set of 6 elements into 3 holes such that there exists a hole with 2 elements. By the pigeonhole principle, we need 4 numbers.

- (25 points) How many relations are there on a set with n elements that satisfies the following properties? Write down your answer and explain the reason.
  (a) Symmetric
  (b) Reflexive and symmetric

  Solution: Consider a set A with n elements and a relation R on set A. Recall that $R \subseteq A \times A$.

  (a) When relation R is symmetric, it contains two types of elements (or pair of elements) from $A \times A$:
   * (a, a) with $a \in A$: n such tuples in $A \times A$
   * both (a, b) and (b, a), with $a, b \in A$ and $a \neq b$: $C(n, 2)$ such tuples in $A \times A$

   Each of these elements (or pair of elements) can be either be in R or not. Thus, there are $2^{n(n-1)/2+n} = 2^{n(n+1)/2}$ symmetric relations.

  (b) When relation R is reflexive, (a, a) with $a \in A$ must be in relation R. Since R is symmetric, (b, a) is in R whenever (a, b) is in R with $a, b \in A$ and $a \neq b$. Thus, we can consider (a, b) and (b, a) as a whole. There are $C(n, 2)$ such pairs of tuples (a, b) and (b, a), and pair can be either in R or not in R. Thus, there are $2^{n(n-1)/2}$ such relations.

- (Bonus 25 points)

(a) Use generating functions to prove Pascal's identity:

$$C(n,r) = C(n-1,r) + C(n-1,r-1)$$

when n and r are positive integers with $r < n$. [Hint: Look at the coefficient of x^r in both sides of $(1+x)^n = (1+x)^{n-1} + x(1+x)^{n-1}$.]

(b) Use generating functions to prove Vandermonde's identity:

$$C(m+n,r) = \sum_{k=0}^{r} C(m,r-k)C(n,k),$$

whenever m, n, and r are nonnegative integers with r not exceeding either m or n. [Hint: Look at the coefficient of x^r in both sides of $(1+x)^{m+n} = (1+x)^m(1+x)^n$.]

Solution: Consider a set A with n elements and a relation R on set A. Recall that $R \subseteq A \times A$.

(a) Construct the following generating functions:

$$A(x) = (1+x)^n = \sum_{r=0}^{n} C(n,r)x^r,$$
$$B(x) = (1+x)^{n-1} = \sum_{r=0}^{n-1} C(n-1,r)x^r,$$
$$xB(x) = x(1+x)^{n-1} = \sum_{r=0}^{n-1} C(n-1,r)x^{r+1} = \sum_{r=1}^{n} C(n-1,r-1)x^r,$$

Note that

$$A(x) = B(x)(1+x).$$

Therefore, we must have $C(n,r) = C(n-1,r) + C(n-1,r-1)$ for all r, n.

(b) Construct the following generating functions:

$$A(x) = (1+x)^{m+n} = \sum_{r=0}^{m+n} a_r x^r = \sum_{r=0}^{m+n} C(m+n,r)x^r,$$
$$B(x) = (1+x)^n = \sum_{r=0}^{n} b_r x^r = \sum_{r=0}^{n} C(n,r)x^r,$$
$$C(x) = (1+x)^m = \sum_{r=0}^{m} c_r x^r = \sum_{r=0}^{m} C(m,r)x^r.$$

Note that

$$A(x) = B(x)C(x) = \sum_{r=0}^{m+n} \left(\sum_{k=0}^{r} b_k c_{r-k} \right) x^r = \sum_{r=0}^{m+n} \sum_{k=0}^{r} C(m, r-k)C(n,k)x^r.$$

Therefore, we must have $C(m+n, r) = \sum_{k=0}^{r} C(m, r-k)C(n,k)$ for all $r = 0, 1, ..., m+n$.