question	1	2	3	4	5	6	7	8	9
score									

MATH 213 Final Exam Sample
Fall 2022

NAME: ______________________

Instructor: M. Zhang

Please answer all *ten* questions.

- Show all work for full credit.
- The number of points for each question is noted.
- You may have 6 single-sided pages of cheat sheets (or equivalent); *no* other informational aids are permitted.
- No calculators (or equivalent) are permitted.
- You may not consult or communicate with anyone except the proctors during the Exam.
- NOTE: failure to abide by the above requirements will result in an Exam score of *zero*.

Good luck!

Answer:

1. Logical statement

   (a) Use logical equivalences to prove the following statements.

      (i) $\neg(p \oplus q)$ and $p \leftrightarrow q$ are equivalent.

      (ii) $\neg(p \to q) \to \neg q$ is a tautology.

      (iii) $(p \to q) \to ((r \to p) \to (r \to q))$ is a tautology.

   (b) Give the negation of the statement

$$\forall n \in \mathbb{N}\ (n^3 + 6n + 5 \text{ is odd} \Rightarrow n \text{ is even}).$$

2. (a) If A is an uncountable set and B is a countable set, must $A - B$ be uncountable?

   (b) Give an example of two uncountable sets A and B such that the difference $A - B$ is

   - finite,
   - countably infinite,
   - uncountable.

3. Let $f_1 : \mathbf{R} \to \mathbf{R}^+$ and $f_2 : \mathbf{R} \to \mathbf{R}^+$. Let $g : \mathbf{R} \to \mathbf{R}$, and $f_1(x)$ and $f_2(x)$ are both $\Theta(g(x))$.

   (a) Prove or disprove that $f_1(x)/f_2(x)$ is $\Theta(1)$.

   (b) Prove or disprove that $f_1(f_2(x))$ is $\Theta(g(g(x)))$.

Solutions: (a) This statement is true. There exist positive real numbers C_1, C_2, C_3, and C_4, and real numbers x_1 and x_2 such that $C_1 g(x) \leq f_1(x) \leq C_2 g(x)$ for all $x \geq x_1$ and $C_3 g(x) \leq f_2(x) \leq C_4 g(x)$ for all $x \geq x_2$. We have that, for all $x \geq \max\{x_1, x_2\}$,

$$C_1/C_4 \leq f_1(x)/f_2(x) \leq C_2/C_3. \tag{1}$$

Therefore, $f_1(x)/f_2(x)$ is $\Theta(1)$.

(a) This statement is true. There exist positive real numbers C_1, C_2, C_3, and C_4, and real numbers x_1 and x_2 such that $C_1 g(x) \leq f_1(x) \leq C_2 g(x)$ for all $x \geq x_1$ and $C_3 g(x) \leq f_2(x) \leq C_4 g(x)$ for all $x \geq x_2$. We have that, for all $x \geq \max\{x_1, x_2\}$,

$$C_1/C_4 \leq f_1(x)/f_2(x) \leq C_2/C_3. \tag{2}$$

Therefore, $f_1(x)/f_2(x)$ is $\Theta(1)$.

(b) This statement is false. Consider a counterexample

$$f_1(x) = \exp(x) \quad \text{and} \quad f_2(x) = 2\exp(x). \tag{3}$$

We have $f_1(x)$ and $f_2(x)$ are both $\Theta(\exp(x))$. However, $f_1(f_2(x)) = \exp(2\exp(x))$ which is not $\Theta(\exp(\exp(x)))$.

4. Let $n \in \mathbb{N}$, $n > 0$. Show

$$\binom{2n}{n+1} + \binom{2n}{n} = \frac{1}{2} \cdot \binom{2n+2}{n+1}.$$

Solution: Using the Pascal's identity, we have

$$\binom{2n}{n+1} + \binom{2n}{n} = \binom{2n+1}{n+1} = \binom{2n+1}{n}.$$

Using the Pascal's identity again:

$$\binom{2n+1}{n} + \binom{2n+1}{n+1} = \binom{2n+2}{n+1}.$$

5. Let a, b, and c be integers. Suppose m is an integer greater than 1 and $ac \equiv bc \pmod{m}$. Prove $a \equiv b \pmod{m/\gcd(c, m)}$.

   Solution: Let $m' = m/\gcd(c, m)$. Because all the common factors of m and c are divided out of m to obtain m', it follows that m' and c are relatively prime. Since $ac \equiv bc (\text{mod } m)$, we have m divides $ac - bc = (a - b)c$, which follows that m' divides $(a - b)c$. Since m' and c are relatively prime, we see that m' divides $a - b$, which leads to $a \equiv b \pmod{m'}$.

6. (a) Solve the recurrence equation $t_n = 2t_{n-1} + n + 2^n$ subject to the initial condition $t_0 = 0$.

   (b) Determine the Θ-class of the function $t(n)$ determined in (a).

   Solutions: (a) Using the theorem on the solutions of linear recurrence equations the general form of the solution is

$$t_n = c_1 + c_2 \cdot n + c_3 \cdot 2^n + c_4 \cdot n \cdot 2^n. \tag{4}$$

   Given the initial condition t_0, from which we obtain $t_1 = 3$, $t_2 = 12$, and $t_3 = 35$ using the recurrence equation. We have $c_1 = -2$, $c_2 = -1$, $c_3 = 2$, and $c_4 = 1$. Therefore. we have

$$t_n = -2 - n + 2^{n+1} + n \cdot 2^n.$$

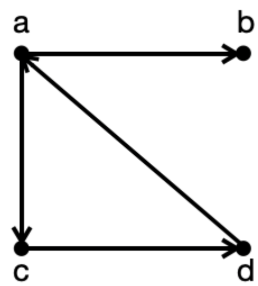
   (b) $t(n) \leq n \cdot 2^n + 2 \cdot 2^n$ and $t(n) \geq n \cdot 2^n$. Therefore, we have $\Theta(n \cdot 2^n)$.

7. Let E_1 and E_2 be equivalence relations on some set A.

   (a) Is $E_1 \cup E_2$ an equivalence relation on A?

   (b) Is $E_1 \cap E_2$ an equivalence relation on A?

8. Consider relation R represented by the following graph.



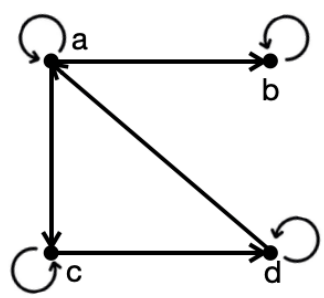
(a) Indicate whether relation R satisfies the properties or not, respectively
- Reflexive
- Symmetric
- Antisymmetric
- Transitive

(b) Draw the reflexive closure of relation R.

Solutions:

(a) No. No. Yes. No.

(b)



9. Consider a relation R defined on the set of functions from $\mathbf{Z}^+$ to $\mathbf{Z}^+$. Consider any of these functions $f : \mathbf{Z}^+$ to $\mathbf{Z}^+$ and $g : \mathbf{Z}^+$ to $\mathbf{Z}^+$, $(f, g) \in R$ if and only if $f(n)$ is $O(g(n))$.

   (a) Is R reflexive? Explain your answer.

   (b) Is R transitive? Explain your answer.

   (c) Prove or disprove R is an equivalence relation.

   (d) Prove or disprove R is a partial ordering.

   Solution:

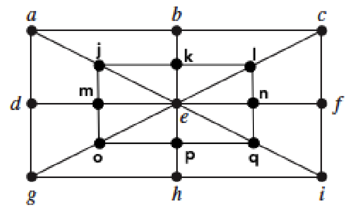
   (a) Yes. Since $f(n) \leq 1f(n)$ for all $n \geq 0$, for any function $f(n)$. Therefore, $f(n)$ is $O(f(n))$ for all function $f(n)$, which implies that R is reflexive.

   (b) Yes. Let $f(n)$ be $O(g(n))$ and $g(n)$ be $O(h(n))$. It is easy to show that $f(n)$ is also $O(h(n))$. Therefore, R is transitive.

   (c) No, because R is not symmetric. To disprove R is symmetric, consider an counterexample with $f(n) = n$ and $g(n) = n^2$. It is easy to see that $(f, g) \in R$ but $(g, f) \notin R$. Therefore, R is not symmetric and hence R is not an equivalence relation.

   (d) No, because R is not antisymmetric, neither. To disprove R is symmetric, consider an counterexample with $f(n) = n$ and $g(n) = 2n$. We see that $(f, g) \in R$ and $(g, f) \in R$, which implies that R is not antisymmetric. Therefore, R is not a partial ordering.

10. Consider the following graph:



   (a) Does the graph contain a Hamilton cycle?

   (b) Does the graph contain an Euler cycle?

   (c) Show that the graph is not bipartite.

   (d) Show that if a single vertex is removed, then the graph becomes bipartite and admits a perfect matching.

Solution:

(a) The cycle $a-d-g-h-i-f-c-b-k-l-n-q-p-o-m-e-j-a$ is a Hamilton cycle, as it contains every vertex exactly once.

(b) The vertices a, b, c all have degree 3, so by Euler's criterium there cannot be an Euler cycle.

(c) There are edges between any two of the three vertices e, j, m. For an arbitrary partition $V = V_1 \cup V_2$, one of the sets V_i must contain at least two of these three vertices. This violates the requirement that in a bipartite graph there cannot exist edges $\{v, w\}$ with $v, w \in V_i$. Hence the graph is not bipartite.

(d) If we omit the vertex e, then we obtain a bipartition $V = V_1 \cup V_2$ with $V_1 = \{a, c, g, i, k, m, n, p\}$ and $V_2 = \{b, d, f, h, j, l, o, q\}$.
Take the subset
$M = \{\{a, j\}, \{b, k\}, \{c, l\}, \{d, m\}, \{f, n\}, \{g, o\}, \{h, p\}, \{i, q\}\} \subset E$
of the set of edges. All edges in M have one endpoint in V_1 and another one in V_2 , and no two edges in M have a common endpoint. Hence M is a matching. For each $v \in V_1$ there is exactly one edge in M that is incident to v, so M is complete for V_1. For each $w \in V_2$ there is also exactly one edge in M that is incident to w, so M is also complete for V_2. Hence M is a perfect matching.

11. (Bonus) Show that $\log_2 3$ is an irrational number. Recall that an irrational number is a real number x that cannot be written as the ratio of two integers.

    Solution: Suppose that $\log_2 3 = p/q$ for some integers p and q. Then, we have $3^q = 2^p$, which is impossible since 3^q is odd and 2^p is even. Hence $\log_2 3$ must be irrational.

12. (Bonus) Prove that $f_k f_n + f_{k+1} f_{n+1} = f_{n+k+1}$ for all nonnegative integers n and k, where f_i denotes the ith Fibonacci number.

    Let $P(n)$ be the statement that

    $$f_k f_n + f_{k+1} f_{n+1} = f_{n+k+1}$$

    holds for all nonnegative integer k. We use the strong principle of induction in the following:

    - Case $n = 0$: $f_n = 0$ and $f_{n+1} = 1$. We have that, for all nonnegative integer k, it follows that

      $$f_k f_n + f_{k+1} f_{n+1} = f_{k+1} = f_{n+k+1}.$$

      Therefore, $P(0)$ is true.
    - Case $n = 1$: $f_n = 1$ and $f_{n+1} = 1$. We have that, for all nonnegative integer k, it follows that

      $$f_k f_n + f_{k+1} f_{n+1} = f_k + f_{k+1} = f_{k+2} = f_{n+k+1}.$$

      Therefore, $P(1)$ is true.
    - Inductive step: Suppose that the statements $P(n)$ and $P(n-1)$ are true. Therefore, for all nonnegative integer k,

      $$f_k f_n + f_{k+1} f_{n+1} = f_{n+k+1},$$
      $$f_k f_{n-1} + f_{k+1} f_n = f_{n+k}.$$

      Since $f_n + f_{n-1} = f_{n+1}$ and $f_n + f_{n+1} = f_{n+2}$ and $f_{n+k+1} + f_{n+k} = f_{n+k+2}$ for all k. We have that, for all k,

      $$f_k f_{n+1} + f_{k+1} f_{n+2} = f_{n+k+2}.$$

      That is, $P(n+1)$ is also true.

    Note that, we need to prove that each statement for each step is true for all k.