question	1	2	3	4	5	6	7	8	9
score									

MATH 213 Final Exam Sample
Fall 2022

NAME: ______________________

Instructor: M. Zhang

Please answer all *ten* questions.

- Show all work for full credit.
- The number of points for each question is noted.
- You may have 6 single-sided pages of cheat sheets (or equivalent); *no* other informational aids are permitted.
- No calculators (or equivalent) are permitted.
- You may not consult or communicate with anyone except the proctors during the Exam.
- NOTE: failure to abide by the above requirements will result in an Exam score of *zero*.

Good luck!

Answer:

1. Logical statement

   (a) Use logical equivalences to prove the following statements.

   (i) $\neg(p \oplus q)$ and $p \leftrightarrow q$ are equivalent.

   (ii) $\neg(p \to q) \to \neg q$ is a tautology.

   (iii) $(p \to q) \to ((r \to p) \to (r \to q))$ is a tautology.

   (b) Give the negation of the statement

$$\forall n \in \mathbb{N}\ (n^3 + 6n + 5 \text{ is odd} \Rightarrow n \text{ is even}).$$

2. (a) If A is an uncountable set and B is a countable set, must $A - B$ be uncountable?

   (b) Give an example of two uncountable sets A and B such that the difference $A - B$ is

   - finite,
   - countably infinite,
   - uncountable.

3. Let $f_1 : \mathbf{R} \to \mathbf{R}^+$ and $f_2 : \mathbf{R} \to \mathbf{R}^+$. Let $g : \mathbf{R} \to \mathbf{R}$, and $f_1(x)$ and $f_2(x)$ are both $\Theta(g(x))$.

   (a) Prove or disprove that $f_1(x)/f_2(x)$ is $\Theta(1)$.

   (b) Prove or disprove that $f_1(f_2(x))$ is $\Theta(g(g(x)))$.

4. Let $n \in \mathbb{N}$, $n > 0$. Show

$$\binom{2n}{n+1} + \binom{2n}{n} = \frac{1}{2} \cdot \binom{2n+2}{n+1}.$$

5. Let a, b, and c be integers. Suppose m is an integer greater than 1 and $ac \equiv bc \pmod{m}$. Prove $a \equiv b \pmod{m/\gcd(c,m)}$.

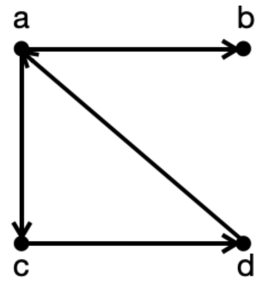
6. (a) Solve the recurrence equation $t_n = 2t_{n-1} + n + 2^n$ subject to the initial condition $t_0 = 0$.
   (b) Determine the Θ-class of the function $t(n)$ determined in (a).

7. Let E_1 and E_2 be equivalence relations on some set A.

   (a) Is $E_1 \cup E_2$ an equivalence relation on A?

   (b) Is $E_1 \cap E_2$ an equivalence relation on A?

8. Consider relation R represented by the following graph.



(a) Indicate whether relation R satisfies the properties or not, respectively
- Reflexive
- Symmetric
- Antisymmetric
- Transitive

(b) Draw the reflexive closure of relation R.

9. Consider a relation R defined on the set of functions from $\mathbf{Z}^+$ to $\mathbf{Z}^+$. Consider any of these functions $f : \mathbf{Z}^+$ to $\mathbf{Z}^+$ and $g : \mathbf{Z}^+$ to $\mathbf{Z}^+$, $(f, g) \in R$ if and only if $f(n)$ is $O(g(n))$.

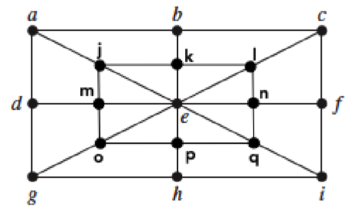
   (a) Is R reflexive? Explain your answer.

   (b) Is R transitive? Explain your answer.

   (c) Prove or disprove R is an equivalence relation.

   (d) Prove or disprove R is a partial ordering.

10. Consider the following graph:



(a) Does the graph contain a Hamilton cycle?

(b) Does the graph contain an Euler cycle?

(c) Show that the graph is not bipartite.

(d) Show that if a single vertex is removed, then the graph becomes bipartite and admits a perfect matching.

11. (Bonus) Show that $\log_2 3$ is an irrational number. Recall that an irrational number is a real number x that cannot be written as the ratio of two integers.

12. (Bonus) Prove that $f_k f_n + f_{k+1} f_{n+1} = f_{n+k+1}$ for all nonnegative integers n and k, where f_i denotes the ith Fibonacci number.