question	1	2	3	4	5	Total
score						

MATH213 First Midterm
Fall 2022

NAME: ______________________

Instructor: M. Zhang

Please answer all first *four* questions (question 5 is optional).

- You are allowed one double-sided cheat sheet.
- Show all work for full credit.
- Each is of equal worth (sub-problems within a problem are of equal worth).
- No calculators are permitted.

Good luck!

Answer:

1. (25 points)

   (a) Suppose P and Q are predicates, and x and y are variables. Suppose all quantifiers we considered have the same nonempty domain. Prove or disprove that $\forall x(P(x) \to Q(x))$ and $\forall x P(x) \to \forall x Q(x)$ are logically equivalent.

   (b) Prove or disprove that, for each real number x, x is rational if and only if $x/2$ is rational.

Blank space for Question 1.

2. (30 points)

   (a) Consider sets A and B. Prove or disprove the following:

   - $\mathcal{P}(A \times B) = \mathcal{P}(B \times A)$.
   - $(A \oplus B) \oplus B = A$, where $A \oplus B$ denotes the set containing those elements in either A or B, but not both.

   (b) Give an example of a function from $\mathbf{N}$ to $\mathbf{N}$ that is

   - one-to-one but not onto.
   - onto but not one-to-one.

Blank space for Question 2.

3. (20 points) Let $f_1 : \mathbf{Z}^+ \to \mathbf{R}^+$, and $f_2 : \mathbf{Z}^+ \to \mathbf{R}^+$. Let $g : \mathbf{Z}^+ \to \mathbf{R}$, and suppose $f_1(x)$ and $f_2(x)$ are both $\Theta(g(x))$.

   (a) Prove or disprove that $(f_1 - f_2)(x)$ is $\Theta(g(x))$.

   (b) Prove or disprove that $(f_1 f_2)(x)$ is $\Theta(g^2(x))$, where $g^2(x) = (g(x))^2$.

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4. (25 points)

   (a) Convert $(11110111)_2$ to an octal expansion.

   (b) Convert $(101)_{10}$ to a binary expansion.

   (c) Compute $\gcd(210, 1638)$ without calculator and explain your answer.

Blank space for Question 4.

5. (Bonus 25 points) Suppose that a is not divisible by the prime p.

   (a) Show that no two of the integers $1 \cdot a, 2 \cdot a, ..., (p-1)a$ are congruent modulo p.

   (b) Use the result in (a), show that

$$(p-1)! \equiv a^{(p-1)}(p-1)!(\mathbf{mod}\ p).$$

Blank space for Question 5.