Midterm 1 – Solutions

Exercise 1. Show that for arbitrary sets A, B, C we have $A - (B \cup C) = (A - B) \cap (A - C)$.

Solution. Let $x \in A - (B \cup C)$ be arbitrary. Then by definition of the difference of sets we have $x \in A$ and $x \notin B \cup C$, hence also $x \notin B$ and $x \notin C$ by the definition of the union of sets. This implies $x \in A - B$ and $x \in A - C$ and further $x \in (A - B) \cap (A - C)$.

Conversely, for an arbitrary $x \in (A - B) \cap (A - C)$ we must have $x \in A$, $x \notin B$ and $x \notin C$, hence also $x \notin B \cup C$. This implies $x \in A - (B \cup C)$.

Exercise 2. Define a binary relation $\preceq$ on $\mathbb{N} \times \mathbb{N}$ by $(i_1, j_1) \preceq (i_2, j_2)$ iff $(i_1 \leq i_2 \wedge j_1 \leq j_2)$. Show the $\preceq$ is a partial order, but not a total order.

Solution. We show that the relation $\preceq$ is reflexive, antisymmetric and transitive.

Reflexivity. For arbitrary $(i, j) \in \mathbb{N} \times \mathbb{N}$ we obviously have $i \leq i$ and $j \leq j$, hence $(i, j) \preceq (i, j)$.

Antisymmetry. Assume $(i_1, j_1) \preceq (i_2, j_2)$ and $(i_2, j_2) \preceq (i_1, j_1)$. Then by definition we have $i_1 \leq i_2$, $j_1 \leq j_2$, $i_2 \leq i_1$, and $j_2 \leq j_1$. The first and the third statement together imply $i_1 = i_2$, and the second and fourth statements imply $j_1 = j_2$, hence together $(i_1, j_1) = (i_2, j_2)$.

Transitivity. Assume $(i_1, j_1) \preceq (i_2, j_2)$ and $(i_2, j_2) \preceq (i_3, j_3)$. Then by definition we have $i_1 \leq i_2$, $j_1 \leq j_2$, $i_2 \leq i_3$, and $j_2 \leq j_3$. The first and the third statement together imply $i_1 \leq i_3$, and the second and fourth statements imply $j_1 \leq j_3$, which together give $(i_1, j_1) \preceq (i_3, j_3)$.

This shows that $\preceq$ is a partial order. As we have $(2, 3) \not\preceq (3, 2)$ and $(3, 2) \not\preceq (2, 3)$, it is not a total order.

Exercise 3. Look at the truth table at the right. Find a propositional formula for φ using propositional atoms p, q, r. Then use the Quine-McCluskey method to simplify the formula.

p	q	r	φ
T	**T**	**T**	**T**
T	**T**	**F**	**F**
T	**F**	**T**	**T**
T	**F**	**F**	**F**
F	**T**	**T**	**T**
F	**T**	**F**	**F**
F	**F**	**T**	**T**
F	**F**	**F**	**T**

Solution. By taking the rows in the truth table with an entry **T** for φ we know that we can write

$$\varphi = (p \wedge q \wedge r) \vee (p \wedge \neg q \wedge r) \vee (\neg p \wedge q \wedge r) \vee (\neg p \wedge \neg q \wedge r) \vee (\neg p \wedge \neg q \wedge \neg r).$$

Thus, we start with the minterms

$$\mu_1 = p \wedge q \wedge r \quad \mu_2 = p \wedge \neg q \wedge r \quad \mu_3 = \neg p \wedge q \wedge r$$
$$\mu_4 = \neg p \wedge \neg q \wedge r \quad \mu_5 = \neg p \wedge \neg q \wedge \neg r$$

We can combine μ_1 with μ_2 and μ_3, μ_2 with μ_4, μ_3 with μ_4, and μ_4 with μ_5, which gives

$$\mu_{1,2} = p \wedge r \quad \mu_{1,3} = q \wedge r \quad \mu_{2,4} = \neg q \wedge r \quad \mu_{3,4} = \neg p \wedge r \quad \mu_{4,5} = \neg p \wedge \neg q$$

Then combine $\mu_{1,2}$ with $\mu_{3,4}$, $\mu_{1,3}$ with $\mu_{2,4}$, which gives $\mu_{1,2,3,4} = r$. Then no more combinations are possible, which means that the algorithm results in

$$\varphi = r \vee (\neg p \wedge \neg q).$$

Exercise 4. Formalise the following statement by a formula in predicate logic:

> There exists a chef of a restaurant with three stars who visits other restaurants with at least one star at least once per month.

Solution. We use predicate symbols *chef* of arity 2 ($\mathit{chef}(c,r)$ means that c is a chef of restaurant r), *restaurant* of arity 2 ($\mathit{restaurant}(r,s)$ means that r is a restaurant r awarded with s stars), *is_month* of arity 1, *visits* of arity 3 ($\mathit{visits}(c,r,d)$ means that c visits the restaurant r on the date d), and *month* of arity 2 ($\mathit{month}(d,m)$ means that the month of date d is m). Furthermore we use $=$ and $\leq$ and natural numbers as constants.

That is, the signature is $\Upsilon = (\mathcal{P}, \mathcal{F})$ with $\mathcal{F} = \mathbb{N}$ and

$$\mathcal{P} = \{\mathit{chef},\ \mathit{restaurant},\ \mathit{is\_month},\ \mathit{visits},\ \mathit{month},\ =, \leq\}\ .$$

Then the desired formula is

$$\begin{aligned}&\exists c.\exists r.\mathit{chef}(c,r) \wedge \mathit{restaurant}(r,3) \wedge \\ &\quad \forall m.\mathit{is\_month}(m) \to \exists r'.\exists s.\mathit{restaurant}(r',s) \wedge \neg(r' = r) \wedge 1 \leq s \wedge \\ &\qquad \exists d.(\mathit{visits}(c,r',d) \wedge \mathit{month}(d,m))\end{aligned}$$

Exercise 5.

Prove by induction that $\displaystyle\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$ holds for all $n \in \mathbb{N}$.

SOLUTION. For $n = 0$ the both sides of the equation are 0, which constitutes the induction base.

For arbitrary n assume $\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$ (induction hypothesis). Then we get

$$\sum_{i=1}^{n+1} \frac{1}{i(i+1)} = \sum_{i=1}^{n} \frac{1}{i(i+1)} + \frac{1}{(n+1)(n+2)} \overset{(i.h.)}{=} \frac{n}{n+1} + \frac{1}{(n+1)(n+2)}$$
$$= \frac{n(n+2)}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)} = \frac{n^2+2n+1}{(n+1)(n+2)} = \frac{(n+1)^2}{(n+1)(n+2)} = \frac{n+1}{n+2} ,$$

which completes the induction step.

EXERCISE 6. Consider a set P of sequences of characters that is inductively defined as the smallest set satisfying the following properties:

(i) The empty sequence ε is an element of P;

(ii) Every sequence of length one with a character in the alphabet $A = \{a, e, h, i, k, m, n, o, p, r, t, u, w\}$ is an element of P;

(iii) Whenever a sequence $w \in P$ and a character $x \in A$ are given, then the composed sequence xwx is an element of P.

Show by structural induction that every sequence of characters $w \in P$ is a palindrome, i.e. $w = w^{-1}$, where w^{-1} is the inverted sequence written backwards from the last character in w to the first.

SOLUTION. For the empty sequence we have $\varepsilon^{-1} = \varepsilon$, and for a sequence consisting of a single character $x \in A$ we also have $x^{-1} = x$, which gives us the base for the structural induction.

Next take an arbitrary $w \in P$, and arbitrary $x \in A$ and assume $w^{-1} = w$ (induction hypothesis). Then $xwx \in P$ and we have $(xwx)^{-1} = xw^{-1}x$. Applying the induction hypothesis gives $(xwx)^{-1} = xw^{-1}x = xwx$, which completes the induction step.

EXERCISE 7. Find all solutions of the following system of linear congruences:

$$x \equiv 4 \mod 5 \qquad x \equiv 2 \mod 8 \qquad x \equiv 2 \mod 3 .$$

SOLUTION. Write the three congruences as $x_i \equiv a_i \bmod n_i$ for $1 \leq i \leq 3$. Then we have $n_1 = 5$, $n_2 = 8$ and $n_3 = 3$, and $a_1 = 4$, $a_2 = 2$, and $a_3 = 2$. As the n_i are pairwise relatively prime, we proceed as in the proof of the Chinese remainder theorem using $m = n_1 n_2 n_3 = 120$ and $m_1 = n_2 n_3 = 24$, $m_2 = n_1 n_3 = 15$, and $m_3 = n_1 n_2 = 40$.

Then m_i is relatively prime to n_i and hence has an inverse in $\mathbb{Z}_{n_i}$. As $m_1 \equiv -1 \bmod n_1$, the inverse $\bar{m}_1$ is -1. As $m_2 \equiv -1 \bmod n_2$, the inverse $\bar{m}_2$ is -1. As $m_3 \equiv 1 \bmod n_3$, the inverse $\bar{m}_3$ is 1.

Then $x = \sum_{i=1}^{3} m_i \bar{m}_i a_i$ is a solution of the system of congruences, i.e.

$$x = -24 \cdot 4 - 15 \cdot 2 + 40 \cdot 2 = -46 \equiv 74 \mod m \ .$$

According to the Chinese remainder theorem solutions to such systems of congruences are unique modulo $m = 120$, so the set of all solutions is $\{74 + 120x \mid x \in \mathbb{Z}\}$.

EXERCISE 8.

Show $2^{n+1} \in O(2^n)$ and $2^{2n} \notin O(2^n)$.

SOLUTION. As $2^{n+1} = c \cdot 2^n$ with $c = 2$ we have $2^{n+1} \in O(2^n)$.

If there exists a constant $c > 0$ and $n_0 \in \mathbb{N}$ with $2^{2n} \leq c2^n$ for all $n > n_0$, we obtain $2^n \leq c$, equivalently $n \cdot \log 2 \leq \log c$ or $n \leq \frac{\log c}{\log 2}$. This cannot be the case, as the right-hand side of this inequality is a constant. Hence $2^{2n} \notin O(2^n)$.

Alternatively, we have $\lim_{n\to\infty} \frac{2^n}{2^{2n}} = \lim_{n\to\infty} \frac{1}{2^n} = 0$, which implies $O(2^n) \subsetneq O(2^{2n})$ and hence $2^{2n} \notin O(2^n)$.