ECE 313
In-Class Activity 4

Write your name and UID here:

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Q1.

(a) Find $P(X > \sqrt{Y})$ if X and Y are jointly distributed with PDF

$$f(x,y) = x + y, \qquad 0 \le x \le 1, 0 \le y \le 1.$$

(b) Find $P(X^2 < Y < X)$ if X and Y are jointly distributed with PDF

$$f(x,y) = 2x, \qquad 0 \le x \le 1, 0 \le y \le 1.$$

a. $P(X > \sqrt{Y}) = \int_0^1 \int_{\sqrt{y}}^1 (x+y)dxdy = \frac{7}{20}$.

b. $P(X^2 < Y < X) = \int_0^1 \int_y^{\sqrt{y}} 2xdxdy = \frac{1}{6}$.

Q2. Let X, Y, and Z be independent uniform(0,1) random variables.

(a) Find $P(\frac{X}{Y} \le t)$ and $P(XY \le t)$.

(b) Find $P\left(\frac{XY}{Z} \le t\right)$.

a.

$$\begin{aligned} P(X/Y \le t) &= \begin{cases} \frac{1}{2}t & t > 1 \\ \frac{1}{2} + (1-t) & t \le 1 \end{cases} \\ P(XY \le t) &= t - t\log t \qquad 0 < t < 1. \end{aligned}$$

b.

$$\begin{aligned} P(XY/Z \le t) &= \int_0^1 P(XY \le zt)dz \\ &= \begin{cases} \int_0^1 \left[\frac{zt}{2} + (1-zt)\right]dz & \text{if } t \le 1 \\ \int_0^{\frac{1}{t}} \left[\frac{zt}{2} + (1-zt)\right]dz + \int_{\frac{1}{t}}^1 \frac{1}{2zt}dz & \text{if } t \le 1 \end{cases} \\ &= \begin{cases} 1 - t/4 & \text{if } t \le 1 \\ t - \frac{1}{4t} + \frac{1}{2t}\log t & \text{if } t > 1 \end{cases}. \end{aligned}$$

Q3. Suppose the distribution of Y, conditional on $X = x$, is $N(x, x^2)$ and the marginal distribution of X is uniform(0,1).

(a) Find E(X), Var(Y) and Cov(X,Y).
(b) Show that whether Y/X and X are independent or not.

a.

$$\begin{aligned}
\mathrm{E}Y &= \mathrm{E}\{\mathrm{E}(Y|X)\} = \mathrm{E}X = \frac{1}{2}. \\
\mathrm{Var}Y &= \mathrm{Var}\left(\mathrm{E}(Y|X)\right) + \mathrm{E}\left(\mathrm{Var}(Y|X)\right) = \mathrm{Var}X + \mathrm{E}X^2 = \frac{1}{12} + \frac{1}{3} = \frac{5}{12}. \\
\mathrm{E}XY &= \mathrm{E}[\mathrm{E}(XY|X)] = \mathrm{E}[X\mathrm{E}(Y|X)] = \mathrm{E}X^2 = \frac{1}{3} \\
\mathrm{Cov}(X,Y) &= \mathrm{E}XY - \mathrm{E}X\mathrm{E}Y = \frac{1}{3} - \left(\frac{1}{2}\right)^2 = \frac{1}{12}.
\end{aligned}$$

b. The quick proof is to note that the distribution of $Y|X = x$ is $\mathrm{n}(1,1)$, hence is independent of X. The bivariate transformation $t = y/x$, $u = x$ will also show that the joint density factors.