ZJUI **SP2024**

ECE 313
In-Class Activity 2
Write your name and UID here:

____

Q1. Let X be a discrete random variable with probability mass function p given by

a	-1	0	1	2
$p(a)$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{2}$

and $p(a) = 0$ for all other a.

a) Let the random variable Y be defined by $Y = X^2$. Calculate the probability mass function of Y.

b) Calculate the value of the distribution functions of X and Y in $a = 1$, $a = 3/4$, and $a = \pi - 3$.

4.2 a Since $\mathrm{P}(Y=0) = \mathrm{P}(X=0)$, $\mathrm{P}(Y=1) = \mathrm{P}(X=-1) + \mathrm{P}(X=1)$, $\mathrm{P}(Y=4) = \mathrm{P}(X=2)$, the following table gives the probability mass function of Y:

a	0	1	4
$p_Y(a)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{1}{2}$

4.2 b $F_X(1) = \mathrm{P}(X \leq 1) = 1/2$; $F_Y(1) = 1/2$; $F_X(\frac{3}{4}) = \frac{3}{8}$; $F_Y(\frac{3}{4}) = \frac{1}{8}$; $F_X(\pi - 3) = F_X(0.1415...) = 3/8$; $F_Y(\pi - 3) = 1/8$.

Q2. You decide to play monthly in two different lotteries, and you stop playing as soon as you win a prize in one (or both) lotteries of at least one million euros. Suppose that every time you participate in these lotteries, the probability to win one million (or more) euros is p_1 for one of the lotteries and p_2 for the other. Let M be the number of times you participate in these lotteries until winning at least one prize. What kind of distribution does M have, and what is its parameter?

Geo(p1+p2-p1p2)

Q4. A group of m people decides to use the elevator in a building of 21 floors. Each of these persons chooses his or her floor independently of the others and completely at random, so that each person selects a floor with probability 1/21. Let S_m be the number of times the elevator stops. In order to study S_m, we introduce for $i = 1,2,\ldots.21$ random variables R_i, given by

$$R_i = \begin{cases} 1 & \text{if the elevator stops at the } i\text{th floor} \\ 0 & \text{if the elevator does not stop at the } i\text{th floor.} \end{cases}$$

a) What is the distribution of R_i?

b) What is the distribution of S_m? Derive the probability mass function for S_1, S_2 and S_3.

4.10 a Each R_i has a Bernoulli distribution, because it can only attain the values 0 and 1. The parameter is $p = \mathrm{P}(R_i = 1)$. It is not easy to determine $\mathrm{P}(R_i = 1)$, but it is fairly easy to determine $\mathrm{P}(R_i = 0)$. The event $\{R_i = 0\}$ occurs when none of the m people has chosen the ith floor. Since they make their choices independently of each other, and each floor is selected by each of these m people with probability 1/21, it follows that

$$\mathrm{P}(R_i = 0) = \left(\frac{20}{21}\right)^m.$$

Now use that $p = \mathrm{P}(R_i = 1) = 1 - \mathrm{P}(R_i = 0)$ to find the desired answer.

4.10 b If $\{R_1 = 0\}, \ldots, \{R_{20} = 0\}$, we must have that $\{R_{21} = 1\}$, so we cannot conclude that the events $\{R_1 = a_1\}, \ldots, \{R_{21} = a_{21}\}$, where a_i is 0 or 1, are independent. Consequently, we cannot use the argument from Section 4.3 to conclude that S_m is $Bin(21, p)$. In fact, S_m is not $Bin(21, p)$ distributed, as the following shows. The elevator will stop at least once, so $\mathrm{P}(S_m = 0) = 0$. However, if S_m would have a $Bin(21, p)$ distribution, then $\mathrm{P}(S_m = 0) = (1 - p)^{21} > 0$, which is a contradiction.

4.10 c This exercise is a variation on finding the probability of no coincident birthdays from Section 3.2. For $m = 2$, $S_2 = 1$ occurs precisely if the two persons entering the elevator select the same floor. The first person selects any of the 21 floors, the second selects the same floor with probability 1/21, so $\mathrm{P}(S_2 = 1) = 1/21$. For $m = 3$, $S_3 = 1$ occurs if the second and third persons entering the elevator both select the same floor as was selected by the first person, so $\mathrm{P}(S_3 = 1) = (1/21)^2 = 1/441$. Furthermore, $S_3 = 3$ occurs precisely when all three persons choose a different floor. Since there are $21 \cdot 20 \cdot 19$ ways to do this out of a total of 21^3 possible ways, we find that $\mathrm{P}(S_3 = 3) = 380/441$. Since S_3 can only attain the values $1, 2, 3$, it follows that $\mathrm{P}(S_3 = 2) = 1 - \mathrm{P}(S_3 = 1) - \mathrm{P}(S_3 = 3) = 60/441$.

Q3. We throw a coin until a head turns up for the second time, where p is the probability that a throw results in a head and we assume that the outcome of each throw is independent of the previous outcomes. Let X be the number of times we have thrown the coin.

a) Determine $P(X = 2)$, $P(X = 3)$ and $P(X = 4)$.
b) Determine $P(X = n)$ for $n \geq 2$.

4.14 a Denoting 'heads' by H and 'tails' by T, the event $\{X = 2\}$ occurs if we have thrown HH, i.e, if the outcome of both the first and the second throw was 'heads'. Since the probability of 'heads' is p, we find that $\mathrm{P}(X = 2) = p \cdot p = p^2$. Furthermore, $X = 3$ occurs if we either have thrown HTH, or THH, and since the probability of throwing 'tails' is $1 - p$, we find that $\mathrm{P}(X = 3) = p \cdot (1 - p) \cdot p + (1 - p) \cdot p \cdot p = 2p^2(1 - p)$. Similarly, $X = 4$ can only occur if we throw $TTHH$, $THTH$, or $HTTH$, so $\mathrm{P}(X = 4) = 3p^2(1 - p)^2$.

4.14 b If $X = n$, then the nth throw was heads (denoted by H), and all but one of the previous $n - 1$ throws were tails (denoted by T). So the possible outcomes are

$$H\underbrace{TT\cdots T}_{n-2 \text{ times}}H, \quad TH\underbrace{TT\cdots T}_{n-3 \text{ times}}H, \quad TTH\underbrace{TT\cdots T}_{n-4 \text{ times}}H, \quad \ldots, \quad \underbrace{TT\cdots T}_{n-2 \text{ times}}HH$$

Notice there are exactly $\binom{n-1}{1} = n - 1$ of such possible outcomes, each with probability $p^2(1 - p)^{n-2}$, so $\mathrm{P}(X = n) = (n - 1)p^2(1 - p)^{n-2}$, for $n \geq 2$.