# ECE 313
## Homework 9 solution

## Problem 1 –

a) E[T] = 0.25 = 1/λ.

Therefore λ = 4.

$f(\mathrm{t}) = 4e^{-4t}$ (in minutes)

b) λ = 4 queries/minutes = 4/60 = 1/15 queries/second = 0.067 queries per second

$$\mathrm{P}(\mathrm{N}_t = \mathrm{i}) = e^{-0.067t}\frac{(0.067\mathrm{t})^{\mathrm{i}}}{i!},\ \mathrm{i}=0, 1, 2, \ldots \quad (t \text{ is in seconds})$$

c) $\mathrm{P}(\mathrm{N}_{10} > 4) = 1 - \mathrm{P}(\mathrm{N}_{10} \leq 4) = 1 - \sum_{i=0}^{4} e^{-0.067\times 10}\frac{(0.067\times 10)^{\mathrm{i}}}{i!}$ = 6.33e-4

d) 2 minutes = 120 seconds

$\mathrm{P}(\mathrm{N}_{120} \leq 5) = \sum_{i=0}^{5} e^{-0.067\times 120}\frac{(0.067\times 120)^{\mathrm{i}}}{i!}$ = 0.10

## Problem 2 – (20 pts)

(a) Observe that $Y$ takes values in the interval $[1, +\infty)$.

$$F_Y(c) = P[\exp(X) \leq c] = \begin{cases} P[X \leq \ln c] = 1 - \exp(-\lambda \ln c) = 1 - c^{-\lambda} & c \geq 1 \\ 0 & c < 1 \end{cases}$$

Differentiate to obtain

$$f_Y(c) = \begin{cases} \lambda c^{-(1+\lambda)} & c \geq 1 \\ 0 & c < 1 \end{cases}$$

(b) Observe that $Z$ takes values in the interval $[0, 3]$.

$$F_Z(c) = P[\min\{X, 3\} \leq c] = \begin{cases} 0 & c < 0 \\ P[X \leq c] = 1 - \exp(-\lambda c) & 0 \leq c < 3 \\ 1 & c \geq 3 \end{cases}$$

**(a) 10 pts (b) 10 pts**

## Problem 3 – (15 pts)

a)

$$f_X(x) = \int_0^\infty e^{-\frac{x}{\alpha}} y e^{-y^2} dy = e^{-\frac{x}{\alpha}} \int_0^\infty y e^{-y^2} dy = e^{-\frac{x}{\alpha}} \left[-\frac{1}{2} e^{-y^2}\right]_0^\infty = \frac{1}{2} e^{-\frac{x}{\alpha}} \text{ (5pts)}$$

$$f_Y(x) = \int_0^\infty e^{-\frac{x}{\alpha}} y e^{-y^2} dx = y e^{-y^2} \int_0^\infty e^{-\frac{x}{\alpha}} dx = y e^{-y^2} \left[-\alpha e^{-\frac{x}{\alpha}}\right]_0^\infty = \alpha y e^{-y} \text{ (5pts)}$$

b) For X and Y to be independent, $f_{X,Y}(x,y) = f_X(x)f_Y(y) = \frac{1}{2}\alpha e^{-\frac{x}{\alpha}} y e^{-y}$
Therefore, $\alpha = 2$ (5pts)

## Problem 4 – (20 pts)

(a) $Z$ takes values in the positive real line. So let $z \geq 0$.

$$\begin{aligned} P[Z \leq z] &= P[\min\{X_1, X_2\} \leq z] = P[X_1 \leq z \text{ or } X_2 \leq z] \\ &= 1 - P[X_1 > z \text{ and } X_2 > z] = 1 - P[X_1 > z]P[X_2 > z] = 1 - e^{-\lambda_1 z}e^{-\lambda_2 z} = 1 - e^{-(\lambda_1+\lambda_2)z} \end{aligned}$$

Differentiating yields that

$$f_Z(z) = \begin{cases} (\lambda_1 + \lambda_2)e^{-(\lambda_1+\lambda_2)z}, & z \geq 0 \\ 0, & z < 0 \end{cases}$$

That is, $Z$ has the exponential distribution with parameter $\lambda_1 + \lambda_2$.
(b) $R$ takes values in the positive real line and by independence the joint pdf of $X_1$ and $X_2$ is the product of their individual densities. So for $r \geq 0$ ,

$$\begin{aligned} P[R \leq r] &= P[\frac{X_1}{X_2} \leq r] = P[X_1 \leq rX_2] \\ &= \int_0^\infty \int_0^{rx_2} \lambda_1 e^{-\lambda_1 x_1} \lambda_2 e^{-\lambda_2 x_2} dx_1 dx_2 \\ &= \int_0^\infty (1 - e^{-r\lambda_1 x_2})\lambda_2 e^{-\lambda_2 x_2} dx_2 = 1 - \frac{\lambda_2}{r\lambda_1 + \lambda_2}. \end{aligned}$$

Differentiating yields that

$$f_R(r) = \begin{cases} \frac{\lambda_1\lambda_2}{(\lambda_1 r+\lambda_2)^2} & r \geq 0 \\ 0, & r < 0 \end{cases}$$

**(a) – 10pts, (b) – 10 pts**

## Problem 5 – (15 pts)

(a) The density must integrate to one, so $c = 4/19$.
(b)

$$f_X(x) = \begin{cases} \frac{4}{19}\int_1^2 (1 + xy)dy = \frac{4}{19}[1 + \frac{3x}{2}] & 2 \leq x \leq 3 \\ 0 & \text{else} \end{cases}$$

$$f_Y(y) = \begin{cases} \frac{4}{19}\int_2^3 (1 + xy)dx = \frac{4}{19}[1 + \frac{5y}{2}] & 1 \leq y \leq 2 \\ 0 & \text{else} \end{cases}$$

Therefore $f_{X|Y}(x|y)$ is well defined only if $1 \leq y \leq 2$. For $1 \leq y \leq 2$:

$$f_{X|Y}(x|y) = \begin{cases} \frac{1+xy}{1+\frac{5}{2}y} & 2 \leq x \leq 3 \\ 0 & \text{for other } x \end{cases}$$

**(a) 5pts, (b) 10 pts**