ECE 313 (Section X)
Homework 9 Solution

Problem 1

(a) We know that $\int_{-\infty}^{\infty} f_X(u)du = \int_0^2 f_X(u)du = 1$.

And $\int_0^2 f_X(u)du = (0.5)(2)(c) = c$.

Thus, $c = \boxed{1}$.

(b) From part (a), we have:

$$f_X(u) = \begin{cases} u, & 0 \le u < 1 \\ -u+2, & 1 \le u < 2 \\ 0, & \text{elsewhere.} \end{cases}$$

Thus, $E[X] = \int_{-\infty}^{\infty} u f_X(u)du = \int_0^2 u f_X(u)du = \int_0^1 uudu + \int_1^2 u(-u+2)du = 1$.

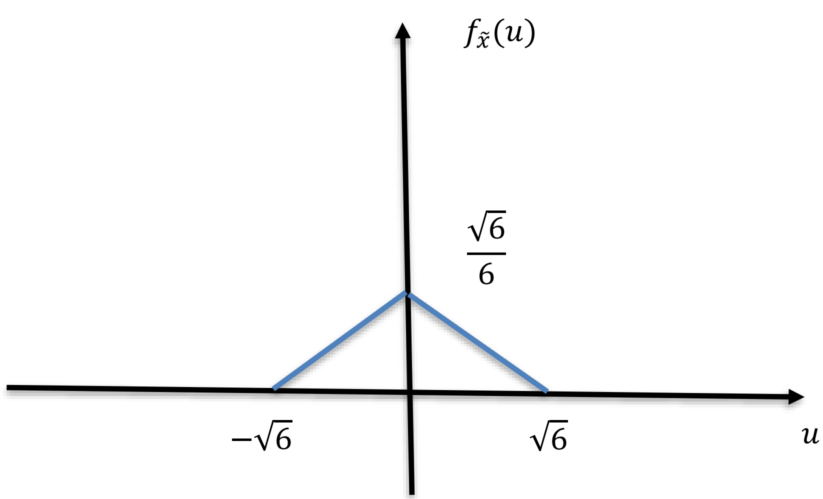
And from the hint we have $Var(X) = \frac{1}{6}$. Then, $\sigma = \sqrt{\frac{1}{6}} = \frac{1}{\sqrt{6}}$

So, $\widetilde{X} = \frac{X-E[X]}{\sigma} = \sqrt{6}X - \sqrt{6}$. And the pdf is,

$$f_{\widetilde{X}}(u) = \frac{1}{\sqrt{6}} f_X\left(\frac{u+\sqrt{6}}{\sqrt{6}}\right).$$

Therefore,

$$f_{\widetilde{X}}(u) = \begin{cases} \frac{u+\sqrt{6}}{6}, & -\sqrt{6} \le u < 0 \\ \frac{-u+\sqrt{6}}{6}, & 0 \le u < \sqrt{6} \\ 0, & \text{elsewhere.} \end{cases}$$



Problem 2

Answer:

(a) $E[X] = \frac{1}{5} * (1 + 2 + 3 + 4 + 5) = 3$ as all outcomes share the same probability.

(b) $Var(X) = E[X^2] - E[X]^2 = \frac{1}{5} * (1^2 + 2^2 + 3^2 + 4^2 + 5^2) - 3^2 = 2$

(c) All outcomes still share the same probability, $E[X] = \frac{1}{5} * (1 + 2 + 3 + 4 + 5) = 3$. However, X and Y are dependent. There is,

$$E[XY] = \frac{1}{5 * 4} * (14 * 1 + 13 * 2 + 12 * 3 + 11 * 4 + 10 * 5) = 8.5$$

Problem 3

(a) $E[Y] = \int_{-\infty}^{\infty} e^X P(X) = \int_{-\infty}^{\infty} e^x f(x) dx = \int_0^{\infty} e^x \lambda e^{-\lambda x} dx = \frac{\lambda e^x e^{-\lambda x}}{1-\lambda} |_0^{\infty} = \boxed{\frac{\lambda}{\lambda - 1}}$.

(b) $X = \ln Y$. And $X \geq 0$.

So, $Y \geq 1$

Thus, the support of the pdf $f_Y(v)$ is $\boxed{\{v \mid v \geq 1\}}$.

(c) We have:

$$\begin{aligned} F_Y(v) &= P\{Y < v\} \\ &= P\{e^X < v\} \\ &= P\{X < \ln v\} \\ &= \int_0^{\ln v} \lambda e^{-\lambda x} dx \\ &= 1 - v^{-\lambda}. \end{aligned}$$

So,

$$\boxed{F_Y(v) = \begin{cases} 1 - v^{-\lambda}, & v \geq 1 \\ 0, & v < 1 \end{cases}}.$$

Problem 4

Answer:

When $0 \leq v \leq 1$, $F_X(v) = \int_0^v 2vdv = v^2$.

Then, we have

$$\begin{aligned} F_X(v) &= P\{X < v\} \\ &= P\{g(U) < v\} \\ &= P\{U < g^{-1}(v)\} \\ &= \int_0^{g^{-1}(v)} \frac{1}{1-0} du \\ &= g^{-1}(v) \\ &= v^2 \end{aligned}$$

Thus, $\boxed{g(v) = \sqrt{v}, 0 \leq v \leq 1}$.

Problem 5

If Part:

i.e. we have $p_X(0) + p_X(c) = 1$.

So, $E[X] = 0 \cdot p_X(0) + c \cdot p_X(c) = c \cdot p_X(c)$.

Thus, $P\{X \geq c\} = p_X(c) = \frac{E[X]}{c}$.

Only If Part:

i.e. we have $P\{X \geq c\} = \frac{E[X]}{c}$.

Let $Y_c = X - c\mathbb{1}_{\{X \geq c\}}$, where

$$\mathbb{1}_{\{X \geq c\}} = \begin{cases} 1, & x \geq c \\ 0, & x < c \end{cases}$$

Note that X is a nonnegative random variable. Then, Y_c is also a nonnegative random variable. (Since for $x < c$, $Y_c = X \geq 0$; and for $x \geq c$, $Y_c = X - c \geq 0$.)

And,

$$\begin{aligned} E[Y_c] &= E[X - c\mathbb{1}_{\{X \geq c\}}] \\ &= E[X] - cE[\mathbb{1}_{\{X \geq c\}}] \\ &= E[X] - c \cdot p_X(X \geq c) \\ &= E[X] - c \cdot \frac{E[X]}{c} \\ &= 0 \end{aligned}$$

Since Y_c is a nonnegative random variable, $E[Y_c] = 0$ means that $Y_c = 0$.

Thus, $X = c\mathbb{1}_{\{X \geq c\}}$. That is,

$$X = \begin{cases} c, & x \geq c \\ 0, & x < c \end{cases}$$

Therefore, $p_X(0) + p_X(c) = p_X(x < c) + p_X(x \geq c) = 1$.