ECE 313
Homework 7 Solution

Problem 1– X ~ EXP(1/120) => $f(x) = \begin{cases} \frac{1}{120} e^{-\frac{x}{120}} & x > 0 \\ 0 & otherwise \end{cases}$

$$F_X(x) = \begin{cases} 1 - e^{-\frac{x}{120}} & x \geq 0 \\ 0 & otherwise \end{cases},$$

Then N_t is Poisson distributed with parameter 1/120t:

$$P(N_t = k) = e^{-\frac{1}{120}t} \frac{\left(\frac{1}{120}t\right)^k}{k!} \quad , k = 0,1,2,\ldots$$

a) Waiting until the second quantum, means that the job would wait at least for a quantum of 100ms:

$$P(X \geq 100) = 1 - P(X < 100) = 1 - F_X(100) = 1 - \left(1 - e^{-\frac{100}{120}}\right) = e^{-\frac{5}{6}} = 0.435$$

b) Probability of a given job to finish in a quantum of 100ms = $P(X \leq 100) = F_X(100)$
So from 800 jobs, 800* $P(X \leq 100) = 452\ jobs$ are expected to finish within the first quantum.

c) Probability of a given job to finish in a quantum of 100ms is equal to the probability of no jobs arriving in an interval of 100ms: :

$$P(X \leq 100) = 1 - P(X > 100) = P(N_{100} = 0)$$
$$= e^{-\frac{1}{120}(100)} \frac{\left(\frac{100}{120}\right)^0}{0!} = e^{-\frac{100}{120}} = 0.435$$

Problem 2 – Let the lifetime of the memory chips produced by the factory be *X*, which is exponentially distributed with parameter λ = 0.1(years)−1. The CDF of *X* is:

$$F_X(x) = \begin{cases} 1 - e^{-0.1x} & x \geq 0 \\ 0 & otherwise \end{cases}$$

So the probability of the memory chip to last for at least 8 years given that it lasted for 5 years is: $P(X \geq 5 + 3 | X \geq 5) = \frac{P(X\geq 8\ and\ X\geq 5)}{P(X\geq 5\ )} = \frac{P(X\geq 8)}{P(X\geq 5\ )} = \frac{e^{-0.8}}{e^{-0.5}} = e^{-0.3}$

The shorter answer would be: By the memory-less property of the exponential distribution, the memory chip after five years is as good as new. So the probability that it lasts at least three more years is equal to: $P(X \geq 3) = e^{-0.3}$.

Problem 3 –

a) $\mathrm{E(X)} = \sum_0^6 np(n) = 0 \times 0.18 + 1 \times 0.28 + \cdots + 6 \times 0.01 = 1.82$

b) $E(X^2) = \sum_0^6 n^2 p(n) = 0 \times 0.18 + 1 \times 0.28 + \cdots + 36 \times 0.01 = 5.22$

$Var(X) = E(X^2) - E(X)^2 = 5.22 - 1.82^2 = 1.9076$

Problem 4 –

a) $P(N_t = k) = \frac{(2t)^k}{k!} e^{-2t}$

b) $f(t) = 2e^{-2t}$

c) The pmf of N for a fixed period of 10 minutes is: $P(N_{10} = k) = \frac{(20)^k}{k!} e^{-20}$, so the expected number of students in 10 minutes would be $\lambda t = 2 \times 10 = 20$.

In other words, $\frac{2 \text{ students}}{1 \text{ minute}} \times \frac{10\ minutes}{10\ minutes} = \frac{20\ students}{10\ minutes}$

d) Given that the time to arrival T (number of minutes until arrival) is exponentially distributed with rate λ, we know the expected value of T would be E[T] = 1/λ. Therefore, the expected number of minutes until the first arrival is ½ = 0.5.

Problem 5 –

$$E[X] = \int_{-\infty}^{\infty} xf(x)dx$$

$$= \int_1^3 x\, 0.5(x-1)dx = \int_1^3 0.5(x^2 - x)dx = \frac{x^3}{6} - \frac{x^2}{4}\Big]_1^3 = \left(\frac{27}{6} - \frac{9}{4}\right) - \left(\frac{1}{6} - \frac{1}{4}\right)$$

$$= \frac{14}{6} = 2.33$$

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f(x)dx$$

$$= \int_1^3 x^2\, 0.5(x-1)dx = \int_1^3 (x^3/2 - x^2/2)dx = \frac{x^4}{8} - \frac{x^3}{6}\Big]_1^3 = \frac{34}{6} = 5.67$$

$$Var[X] = E[X^2] - E[X]^2 = 5.67 - 2.33^2 = 0.24$$

Problem 6 –

a) To find the pdf of a function of a random variable, we first start by finding the CDF:

$$F_Y(v) = P\{Y \le v\} = P\{-lnX \le v\}$$

$$= P\{X \ge e^{-v}\} = \int_{e^{-v}}^{1} dx = 1 - e^{-v}$$

Then to find the pdf, simply differentiate w.r.t v (for $v > 0$):

$$f_Y(v) = \frac{d}{dv}(1 - e^{-v}) \Rightarrow f_Y(v) = \begin{cases} e^{-v}: & v > 0 \\ 0: & otherwise \end{cases}$$

b) $E[Y] = \int_{-\infty}^{\infty} yf(y)dy = \int_0^{\infty} ye^{-y}dy = -ye^{-y}\Big|_0^{\infty} + \int_0^{\infty} e^{-y}dy = 1$

c) $E[Y] = E[-lnX] = \int_{-\infty}^{\infty} -lnxf(x)dx = \int_{0}^{1} -lnx.1dx = (-xlnx + x)\Big|_{0}^{1} = 1$