ECE 313
Homework 5 Solution

Problem 1 (10 points) –

(5 points each)

a) P (X >= 15.3) = P (Z >= (15.3 – 15)/0.5) = P(Z >= 0.6) = 1 – 0.7257 = 0.2743

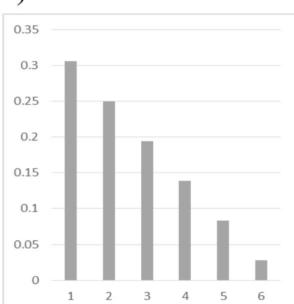
b) 1 - P(-1.6 <= Z <= 1.6) = 2 * (1 - P(Z <= 1.6)) = 2 * (1 – 0.9452) = 0.1096

Problem 2(10 points) –

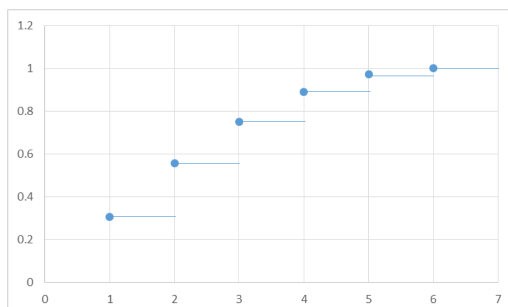
X	P(X)
1	(6+6-1) / 36 = 0.31
2	(5+5-1) / 36 = 0.25
3	(4+4-1) / 36 = 0.19
4	(3+3-1) / 36 = 0.14
5	(2+2-1) / 36 = 0.08
6	(1+1-1) / 36 = 0.03

X	$F_X(x)$
0 ($x \le 0$)	0
1 ($0 < x \le 1$)	0 + 0.31 = 0.31
2 ($1 < x \le 2$)	0.31 + 0.25 = 0.56
3 ($2 < x \le 3$)	0.56 + 0.19 = 0.75
4 ($3 < x \le 4$)	0.75 + 0.14 = 0.89
5 ($4 < x \le 5$)	0.89 + 0.08 = 0.97
6 ($5 < x \le 6$)	0.97 + 0.03 = 1.00
$6 < x$	1

a)



b)



c) $P\{X \le 1\} = P(X=1) = F_x(1) = 0.31$

d) $P\{X < 1\} = 0$

e) $P\{X \ge 6\} = 1 - P(X < 6) = 1 - [P(X = 1) + \dots + P(X = 5)] = 1 - F_x(5) = P(X = 6) = 0.03$

f) $P\{X = 6\} = 0.03$

g) $P\{X > 6\} = 0$

h) $P\{3 \le X \le 6\} = P(X=3) + P(X=4) + P(X=5) + P(X=6) = 0.44$

i) $P\{|X - 3| \le 0.1\} = P\{2.9 \le X \le 3.1\} = P(X \le 3.1) - P(X < 2.9) = P(3) = 0.19$

Problem 3

A) X is binomial random variable with p = 6/100, so the PMF is:

$$P(X = n) = \binom{10}{X} * \left(\frac{6}{100}\right)^X \left(\frac{94}{100}\right)^{10-X}$$

B) $P(X = 0) = 0.94^{10} = 0.5386$

C) $P(X = 1) = 10 * 0.06 * 0.94^9 = 0.3438$

D) $P(X > 2) = 1 - P(X = 0) - P(X = 1) - P(X = 2) = 1-0.5386-0.3438-0.09875 = 0.01885$

Problem4 (15 points) –

a) (5 points) From the definition of CDF for a continuous random variable, we know that: $P(a < X < b) = \int_a^b f(x)dx$. So $P(X > 20)$ can be calculated in either of the following ways:

1. $P(X > 20) = 1 - P(X \le 20) = 1 - \int_{-\infty}^{20} f(x)dx = 1 - \int_{-\infty}^{10} 0\,dx - \int_{10}^{20} \frac{10}{x^2}dx$

$$= 1 - \left[-\frac{10}{x}\right]_{10}^{20} = 1 - \left(\frac{10}{20} - \frac{10}{10}\right) = 1 - \frac{1}{2} = \frac{1}{2}$$

2. $P(X > 20) = \int_{20}^{\infty} f(x)dx = \left[-\frac{10}{x}\right]_{20}^{\infty} = \left(-\frac{10}{\infty} + \frac{10}{20}\right) = \frac{1}{2}$

b) (10 points) The CDF of X can be derived from its pdf as follows:

$$F(x) = \int_{-\infty}^{x} f(y)\,dy$$

For $x \le 10, f(x) = 0$ so we have: $F(x) = 0$

For $x > 10$, we have: $F(x) = \int_{10}^{x} \frac{10}{y^2}dy = \left[-\frac{10}{y}\right]_{10}^{x} = -\frac{10}{x} + 1$

$$F(x) = \begin{cases} 0 & x \le 10 \\ 1 - \dfrac{10}{x} & x > 10 \end{cases}$$

c) (5 points) We first find the probability of a device functioning for at least 15 hours:

$$P(X > 15) = \int_{15}^{\infty} f(x)dx = \left[-\frac{10}{x}\right]_{15}^{\infty} = \frac{2}{3}$$

If we let Y be the number of devices that will function for more than 15 hours, then the probability of at least 2 out of 5 devices to function properly for at least 15 hours can be calculated as follows:

$$P(X \ge 2) = 1 - P(X = 1) - P(X = 0)$$

$$= 1 - \binom{5}{1}\left(\frac{2}{3}\right)^1\left(\frac{1}{3}\right)^4 - \binom{5}{0}\left(\frac{2}{3}\right)^0\left(\frac{1}{3}\right)^5$$

$$= 1 - \frac{11}{243} = \frac{232}{243}$$

Problem 5 –

a) If tested disks are not marked and are not eliminated from future tests, *N* can be modeled as a geometric random variable with parameter *p*. In each trial, the test program would select the defective disk with probability $p = \left(\frac{1}{n}\right)$ and an non-defective disk with the probability $q = 1 - p = \left(\frac{n-1}{n}\right)$. So the probability mass function of *N* can be written as follows:

$P\{N = 1\} = p(1) = p = 1/n$

$$P\{N=2\} = p(2) = (1-p)p = \left(\frac{n-1}{n}\right)\left(\frac{1}{n}\right)$$
$$\vdots$$
$$P\{N=k\} = p(k) = (1-p)^{k-1}p = \left(\frac{n-1}{n}\right)^{k-1}\left(\frac{1}{n}\right)$$

b) If tested disks are eliminated after each trial, the probability of finding the defective disk changes at each trial. In the first trial, the program would select the defective disk with probability $p_1 = \frac{1}{n}$ and a non-defective disk with the probability $q_1 = 1 - p_1 = \left(\frac{n-1}{n}\right)$., in the second trial, given that the previous tested disk was not defective, the defective disk is selected with probability of $p2 = \left(\frac{1}{n-1}\right)$ and a non-defective disk will be selected with probability $q_2 = 1 - p_2 = \left(\frac{n-2}{n-1}\right)$, and so on.

So the probability mass function for *N* can be written as follows:

$$P\{N=1\} = p(1) = p_1 = 1/n$$
$$P\{N=2\} = p(2) = (1-p_1)p_2 = \left(\frac{n-1}{n}\right)\left(\frac{1}{n-1}\right) = \frac{1}{n}$$
$$P\{N=3\} = p(3) = (1-p_1)(1-p_2)p_3 = \left(\frac{n-1}{n}\right)\left(\frac{n-2}{n-1}\right)\left(\frac{1}{n-2}\right) = \frac{1}{n}$$
$$\vdots$$
$$P\{N=k\} = p(k) = (1-p_1)(1-p_2)\dots(1-p_{k-1})p_k$$
$$= \left(\frac{n-1}{n}\right)\left(\frac{n-2}{n-1}\right)\left(\frac{n-3}{n-2}\right)\dots\left(\frac{1}{n-k+1}\right) = \frac{1}{n}$$

Problem 6 –

a) The loop condition is TRUE when the randomly generated number is either 7, 8, 9 or 10. There are 10 possible integers that can be generated by *randInt()*.
Therefore $p = 4/10 = 0.4$

b) i. If the first evaluation of the condition fails, then S would never be executed.

Hence, X = {0, 1, 2, …}

ii. For a do while loop, the body of the loop is executed before evaluation of the condition. Therefore S is executed at least one time.
Hence, X = {1, 2, 3, …}

c) i. P (X = 0) = 1-p, P(X=1) = p(1-p), P(X=2) = p^2(1-p) and so on ...
Therefore the pmf of *X* can be written as: $p_x(i) = (1-p)p^i\,, for\ i = 0,1,2, ...$
which shows *X* is a modified geometric distribution with parameter 1-p.

ii. P (X = 1) = p, and P (X = 2) = p(1-p), P(X = 3) = p$(1-p)^2$, and so on ...
Therefore the pmf of *X* can be written as: $p_x(i) = p(1-p)^{i-1}\,, for\ i = 1,2,3, ...$
which shows *X* is a geometric distribution with parameter p.