ECE 313
Homework 4 Solution

Problem 1 –

a) T is a continuous random variable.

b) X and N are a discrete random variable. Solution for pmf given in problem set.

c) Y is a discrete random variable. You can easily construct a table for each possible outcome in the experiment and assign the values of Y accordingly:

If Y represent the difference between the number of heads and the number of tails obtained when a fair coin is tossed **3 times,** the outcomes of the experiment would be:

Toss 1	Toss 2	Toss 3	Y
H	H	H	3
H	H	T	1
H	T	H	1
H	T	T	1
T	H	H	1
T	H	T	1
T	T	H	1
T	T	T	3

Since this is a fair coin, each outcome is equally likely. So,

$$P\{X = i\} = \begin{cases} \frac{6}{8} = \frac{3}{4}, & i = 1 \\ \frac{2}{8} = \frac{1}{4}, & i = 3 \\ 0, & else \end{cases}$$

d) L is a continuous random variable.

e) X is a discrete random variable, because it takes values on a finite set of values. The player can:

i. Loose all rounds ($0)
ii. Win one of the three rounds ($1000)
iii. Win two rounds ($5000)
iv. Win all three rounds ($6000)

Hence, X = {0, 1000, 5000, 6000} with the following probabilities,

$$P(i) = (0.45)(0.3)(0.6) = 0.081$$

$$P(ii) = (0.55)(0.3)(0.6) + (0.45)(0.7)(0.6) + (0.45)(0.3)(0.4) = 0.342$$

$$P(iii) = (0.55)(0.7)(0.6) + (0.55)(0.3)(0.4) + (0.45)(0.7)(0.4) = 0.423$$

$P(iv) = (0.55)(0.7)(0.4) = 0.154$

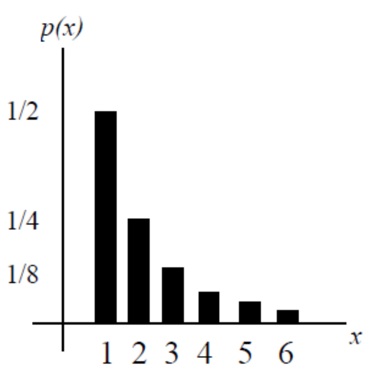
f) W is a discrete random variable.
W takes the values of {0, 1, 2, 3, 4, 5, 6, 7} with the following probabilities:
$P\{X = 0\} = p(0) = \binom{7}{0}(0.15)^7(0.85)^0$
$P\{X = 1\} = p(1) = \binom{7}{1}(0.15)^6(0.85)^1$
$\vdots$
$P\{X = k\} = p(k) = \binom{7}{k}(0.15)^{7-k}(0.85)^k$ $k = 0, 1, 2, 3, 4, 5, 6, 7$

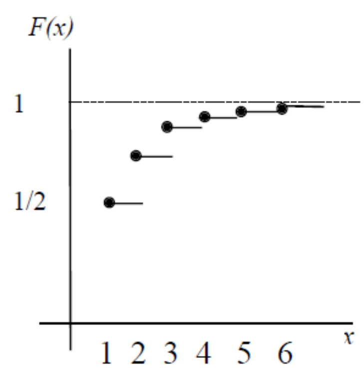
g) A is a continuous random variable.

h) X is a discrete random variable.
The student will get a correct answer to each question, by the probability 1/3.
X is the number of correct answers that the student might get.
So it takes the values of {0, 1, 2, 3, 4, 5} with the following probabilities:
$P\{X = 0\} = p(0) = \binom{5}{0}(2/3)^5(1/3)^0$
$P\{X = 1\} = p(1) = \binom{5}{1}(2/3)^4(1/3)^1$
$\vdots$
$P\{X = k\} = p(k) = \binom{5}{k}(2/3)^{5-k}(1/3)^k$ $k = 0, 1, 2, 3, 4, 5$

Problem 2 –

a) The pmf $P(X = i) = 2^{-i}$ (for $i \geq 1$) can be drawn as follows:





b) For $p(x = i) = 2^{-i}$ to be a valid pmf, the following two conditions must be satisfied:
1) $0 \leq p(x = i) \leq 1$ for all valid i ($i = 1, 2, \ldots$)

Which holds because according to the figure 2^{-i} is decreasing, and:

$1/2 \geq \left(\frac{1}{2}\right)^i \geq 0 \; for \; all \; i = 1,2, \ldots$

2) $\sum_{x=1}^{\infty} p(x) = 1$:

Which holds because: $\sum_{x=1}^{\infty} 2^{-i} = \frac{1}{2}\sum_{x=1}^{\infty}\left(\frac{1}{2}\right)^{i-1} = \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1$

c) $P(X \leq 4) = F_x(4) = P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4)$
$= 1/2 + 1/4 + 1/8 + 1/16 = 15/16$.

d) $P(X > 4) = 1 - P(X \leq 4) = 1 - 15/16 = 1/16$.

e) $P(X < 1) = 0$

f) $P(|X-5| \leq 0.1) = P(4.9 \leq X \leq 5.1) = P(X \leq 5.1) - P(X < 4.9)$
$= F_x(5) - F_x(4) = 31/32 - 15/16 = 0.03125$

g) $\sum_{i=6}^{\infty} P(X = i) = P(X \geq 6) = 1 - P(X < 6) = 1 - P(X \leq 5)$
$= 1 - \frac{1}{2} - \frac{1}{4} - \frac{1}{8} - \frac{1}{16} - \frac{1}{32} = \frac{1}{32}$

Problem 3 –

(a) Invalid, because not all values of $f(x) \geq 0$ for $0 \leq x < 1$

(b) Invalid, because F(0) > 1. Also, F is decreasing.

(c) Invalid, because $f(x) < 0$ for $-1 \leq x < 0$

Problem 4 –

a)

Solution: Note that $T \sim$ Exponential(0.05).

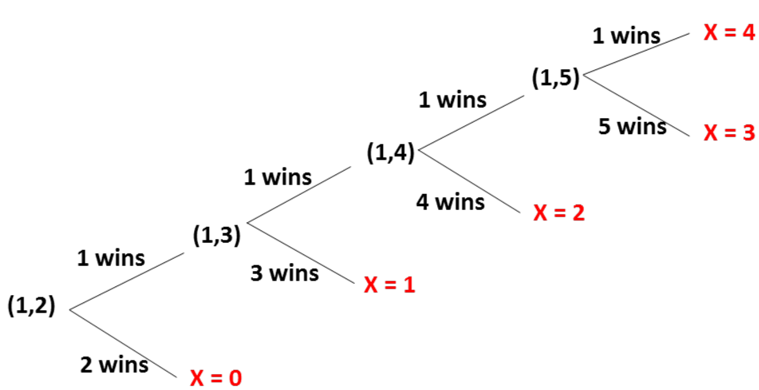
$$F_T(t) = \int_0^t \lambda e^{-\lambda\tau}\, d\tau$$
$$= \frac{\lambda e^{-\lambda\tau}}{-\lambda}\Bigg|_0^t = \begin{cases} 1 - e^{-\lambda t} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

b)

Solution: Note that the exponential distribution is memoryless, e.g., if T is exponential then

$$\begin{aligned} P\{10 \leq T \leq 30 \mid T \geq 10\} &= P\{0 \leq T \leq 20\} \\ &= F_T(20) - F_T(0) \\ &= 1 - e^{(-20/20)} - (1 - e^0) \\ &= 1 - e^{-1} \end{aligned}$$

Problem 5 –



There are 5! possible cases for numbers being distributed to 5 players.

X = 0: For player1 to win 0 times, player1 has to lose to player2.
Therefore, player1 < player2, remaining players can have any combinations.

- If player1 got #1, then obviously loses to player2: 4!
- If player1 got #2, then player2 had 3 possibilities to win: {#3, #4, #5},
  Others had 3! possible combinations: 3 x 3!
- If player1 got #3, then player2 had 2 possibilities: 2 x 3!
- If player1 got #4, then player2 had to have 5: 3!

So, we have $P\{X = 0\} = \frac{4! + 3 \times 3! + 2 \times 3! + 1 \times 3!}{5!} = 0.5$

X = 1: For player1 to win 1 times, player1 has to win player2, but lose to player 3.
Therefore, player2 < player1 < player3, remaining can have any combinations.
For player1 to win once, there are 3 possibilities: {#2, #3, #4}

- If player1 got #2, then player2 should have had {#1}
  and player3 had 3 possibilities to win {#3, #4, #5}
  Others had 2! possible combinations: 1 x 3 x 2!
- If player1 got #3, then player2 had 2 possibilities {#1, #2}
  and player3 had 2 possibilities to win {#3, #4}
  Others had 2! possible combinations: 2 x 2 x 2!
- If player1 got #4, then player2 had 3 possibilities {#1, #2, #3}
  and player3 should have had {#5}
  Others had 2! possible combinations: 3 x 1 x 2!

So, we have $P\{X = 1\} = \frac{3\times2!+2\times2\times2!+3\times1\times2!}{5!} = 0.167$

Similarly,

X = 2: player3, player2 < player1 < player4, remaining does not matter
$P\{X = 2\} = \frac{2!\times2!+3!}{5!} = 0.083$

X = 3: player1 gets 4, player5 gets 5 and remaining does not matter
$P\{X = 3\} = \frac{3!}{5!} = 0.05$

X = 4: player1 gets 5 remaining does not matter
$P\{X = 4\} = \frac{4!}{5!} = 0.2$