ECE 313
Homework2 Solution

Problem 1

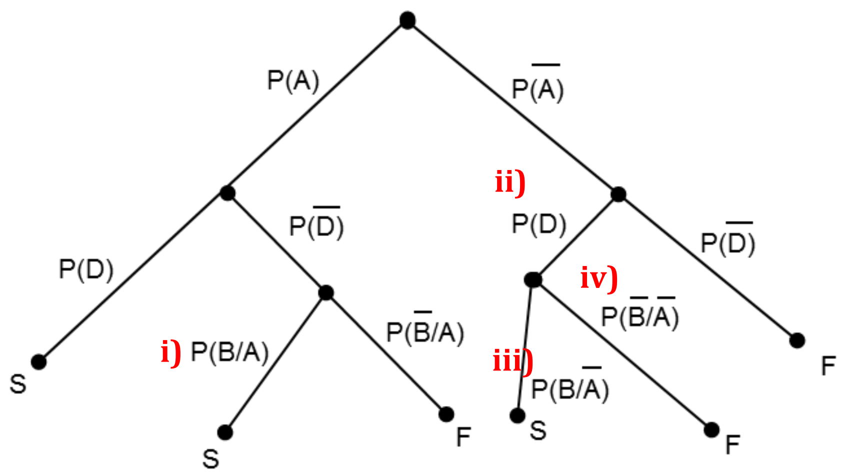
a) i) Given that the detection has failed, i.e., detection test claims $\overline{A}$, for the cooling system to succeed, the backup pump should operate. Therefore, $P(B|A)$. Note, it is A not $\overline{A}$

ii) Detection succeeds (D) or fails ($\overline{D}$)

Now that the primary pump has failed and the test has detected correctly,

iii) The cooling system succeeds when the backup pump functions correctly. Therefore $P(B|\overline{A})$

iv) Otherwise, if the backup pump fails as well, then the SW fails. Therefore $P(\overline{B}|\overline{A})$



b) There are 3 cases where the cooling system fails:

- First is when the detection fails (thinks that the primary pump failed) and the backup pump fails as well. The probability for this event is $P(A)P(\overline{D})P(\overline{B}|A)$
- Second is when the primary pump fails, iCOM detects failure but backup pump fails. The probability for this event is $P(\overline{A})P(D)P(\overline{B}|\overline{A})$
- Third is when the primary pump fails and iCOM did not detect such failure. As the detection failed and thought the primary pump functioned correctly, it would not switch to the backup pump, and would result to a failure. $P(\overline{A})P(\overline{D})$

The probability of failure can be derived by the sum of probabilities of these 3 events:

$$
\begin{aligned}
P(\text{failure}) &= P(\overline{B} \mid A)\, P(\overline{D})\, P(A) + P(\overline{B} \mid \overline{A})\, P(D) P(\overline{A}) + P(\overline{A})\, P(\overline{D}) \\
&= P(\overline{B} \cap A)\, P(\overline{D}) + P(\overline{B} \cap \overline{A})\, P(D) + P(\overline{A})\, P(\overline{D}) \\
&= P((\overline{B} \cap A) \cup \overline{A})\, P(\overline{D}) + P(\overline{B} \cap \overline{A})\, P(D) \\
&= P((\overline{B} \cup \overline{A}) \cap (A \cup \overline{A})) P(\overline{D}) + P(\overline{B} \cap \overline{A})\, P(D) \\
&= P(\overline{B} \cup \overline{A})\, P(\overline{D}) + P(\overline{B} \cap \overline{A})\, P(D)
\end{aligned}
$$

Distributive Law (E3) (applies to the third and fourth lines)

Problem 2 – We will first establish some relationships between events F, E, and G to show the desired properties.

a) We are given that E is independent of F, and that E is independent of G. Stated another way,

$$P(E \cap F) = P(E)P(F)$$

$$P(E \cap G) = P(E)P(G)$$

We want to show or disprove that E is always independent of $F \cup G$; that is,

$$P(E \cap (F \cup G)) = P(E)P(F \cup G)$$

Rewriting this equation gives us:

$$P((E\cap F) \cup (E\cap G))= P(E).[P(F)+P(G)-P(F\cap G)]$$

$$P(E\cap F)+P(E\cap G)-P(E\cap F\cap G)= P(E)P(F)+P(E)P(G)-P(E)P(F\cap G)$$

$$P(E)P(F) + P(E)P(G) - P(E \cap F \cap G) = P(E)P(F) + P(E)P(G) - P(E)P(F \cap G)$$

$$P(E \cap F \cap G) = P(E)P(F \cap G)$$

Thus, <u>E is independent of $F \cup G$ only if E is independent of $F \cap G$.</u>

b) If $F \cap G = \varnothing$, then the previous equation (the condition for independence of E and $F \cup G$) reduces to:

$$P(E \cap \varnothing) = P(E)P(\varnothing)$$

$$0=0$$

which is always true. Thus, <u>E is always independent of $F \cup G$ when F and G are disjoint.</u>

Problem 3 – Using the information provided in the problem description, we can write the following probabilities for the events I that a person is an independent, L that the person is a liberal, C that the person is a conservative, and V that the person voted.

$$P(I) = 0.44,\ P(L) = 0.26,\ P(C) = 0.3$$
$$P(V \mid I) = 0.55,\ P(V \mid L) = 0.38,\ P(V \mid C) = 0.32$$

a) We want to calculate $P(I \mid V)$. From Bayes' formula,

$$P(I|V) = \frac{P(V|I)P(I)}{P(V|I)P(I) + P(V|L)P(L) + P(V|C)P(C)}$$
$$= \frac{(0.55)(0.44)}{(0.55)(0.44) + (0.38)(0.26) + (0.32)(0.3)} = 0.554$$

b) Similar reasoning can be used to show $P(L|V) = 0.226$

c) And that $P(C|V) = 0.220$.

d) Finally, $P(V) = P(V|I)P(I) + P(V|L)P(L) + P(V|C)P(C) = 0.4368$

Problem 4 – Let C' be the event that the previous output was 1, and let C be the event that the current output is 1. From the behavior of the AND gate, we have C' = A' ∩ B' and C = A ∩ B (where A' and B' are the events that the inputs on line A and B were previously 1). By the independence of A' and B', and of A and B, we have P(C') = P(A')P(B') = $\left(\frac{2}{5}\right)\left(\frac{2}{5}\right) = \frac{4}{25}$ and P(C) = P(A)P(B) = $\left(\frac{2}{5}\right)\left(\frac{2}{5}\right) = \frac{4}{25}$. It is also easy to show that events C and C' are independent since events A, A', B, and B' are mutually independent.

We want to find P(switch). The switching is the union of two disjoint events $[C \cap \overline{C'}]$ and $[\bar{C} \cap C']$. So we have:

$$\begin{aligned} P(switch) &= P(C|\overline{C'})P(\overline{C'}) + P(\bar{C}|C')P(C') \\ &= P(C)P(\overline{C'}) + P(\bar{C})P(C') \\ &= \left(\frac{4}{25}\right)\left(\frac{21}{25}\right) + \left(\frac{21}{25}\right)\left(\frac{4}{25}\right) = \frac{168}{625} \end{aligned}$$

Problem 5 – Let S be the event that a black ball is transferred to Box B, and let T be the event that the ball drawn from Box B is black. We can establish the conditional probabilities of event T based on the information contained in the problem description:

$$P(T|S) = \frac{5}{8}$$

$$P(T|\bar{S}) = \frac{4}{8}$$

We are interested in P(T), the probability that a black ball is drawn from Box B. We can use the theorem of total probability to calculate this:

$$P(T) = P(T|S)P(S) + P(T|\bar{S})P(\bar{S}) = \frac{5}{8}.\frac{5}{9} + \frac{4}{8}.\frac{4}{9} = \frac{41}{72}$$

Problem 6

Let W be the event that a white ball is selected (the final outcome of the experiment). Let N_1 and N_2 be the number of balls in the first and second urn, respectively, and let W_1 and W_2 be the number of white balls in the first and second urn, respectively. Let A be the event that you choose to select a ball from the first urn, and B be the event that you choose to select a ball from the second urn. Note that since you are making this selection at random, P(A) = P(B) = 1/2.

By the theorem of total probability, P(W) = P(W | A)P(A) + P(W | B)P(B) since A and B are mutually exclusive and collectively exhaustive. Now,

$$P(W|A) = \frac{W_1}{N_1}, P(W|B) = \frac{W_2}{N_2}$$

So, we are trying to maximize the sum:

$$P(W) = \left(\frac{W_1}{N_1}\right)\left(\frac{1}{2}\right) + (\frac{W_2}{N_2})(\frac{1}{2})$$

subject to the following constraints:

$W_1 \leq N_1$, $W_2 \leq N_2$, $N_1 \geq 1$, $N_2 \geq 1$, $W_1 + W_2 = 10$, $N_1 + N_2 = 20$

We can reduce this to an equation in two unknowns as follows:

$$P(W) = \left(\frac{W_1}{N_1}\right)\left(\frac{1}{2}\right) + \left(\frac{10 - W_1}{20 - N_1}\right)\left(\frac{1}{2}\right)$$

Maximizing this equation is equivalent to maximizing:

$$P(W) = \frac{W_1}{N_1} + \frac{10 - W_1}{20 - N_1}$$

We first seek to maximize the first term, and then maximize the second term given that the first has been maximized. Under the given constraints, W_1/N_1 will be maximized when $W_1 = N_1$.

Thus, the second term becomes $(10 – W_1)/(20 – W_1)$, where W_1 can take on integer values in the range [1, 10] ($W_1 \neq 0$ because we have established that $W_1 = N_1$ and that $N_1 \geq 1$). You can convince yourself that the second term is maximized when $W_1 = 1$.

So, our breakdown is 1 white ball and no black balls in the first urn, and 9 white balls and 10 black balls in the second urn. This gives us:

$$P(W) = \left(\frac{1}{2}\right)\left(1 + \frac{9}{19}\right) = \frac{14}{19}$$