ECE 313
Homework 1 Solution

Problem 1 (20 points – 5 points for each part) –

a. We enter the *for* loop 25 times, and in each loop entry we have 2 possible cases based on the value of variable *p*:
Sample space consists of 2^{25} combinations of 'A' s and 'B' s.
Number of ways exactly 6 'A' s can occur: 25C6

b) There are three positions to fill: the units, the tens, and the hundreds. For the number to be even, there are 5 options for the units position: {0, 2, 4, 6, 8}:

- If the unit position is filled with a non-zero number {2, 4, 6, 8},
  - For the hundreds position, we cannot use 0 and we cannot use the number that was used in units position. So we are left with 8 choices.
  - For the tens position, we cannot use the numbers used in the hundreds or the units positions, so again we are left with 8 choices.
  - Hence, we have 8×8×4 choices = 256.
- If the unit position is filled with zero {0}:
  - For the hundreds position, we are left with 9 choices {1, 2, 3,..., 9}.
  - For the tens position, we cannot use the numbers used in the hundreds or the units positions, so again we are left with 8 choices.
  - Hence, we have 9×8×1 choices = 72.

So the sample space size will be 256 + 72 = 328.

c) The digit 8 can be any of these positions: units, tens, hundreds or thousands. So we have 4 options. If we put '6' on units, tens or hundreds, the two remaining positions can be filled in 9 × 9 ways. Also, there are 4 options (0,1,2,3,4,5) to fill the position of thousands. Therefore, there are 3 x 9 x 9 x 6 = 1458 such numbers. If we put '6' on thousands, the three remaining position can be filled in 9 * 9 * 9 ways. The total number is 1458 + 729 = 2187

(6, 9, 9, 1) - 486
(6, 9, 1, 9) - 486
(6, 1, 9, 9) - 486
(1, 9, 9, 9) - 729

d) - First student has n options for choosing the first book to borrow, n-1 for the second book and n-2 for the third book. Therefore first student has $\binom{n}{3}$ options.

- As the first student has rented 3, second student now has $\binom{n-3}{3}$ options
- Now that 6 of the *n* books are already borrowed, the third student has $\binom{n-6}{3}$ options left.
- Now that 9 of the n books are already borrowed, the third student has $\binom{n-9}{3}$ options left.

Therefore, the sample size is $\binom{n}{3} \times \binom{n-3}{3} \times \binom{n-6}{3} \times \binom{n-9}{3}$.

Problem 2 (20 points) – M – 'malicious', B – 'benign'

The sample space for this experiment consists of twelve items, each either being M or B. Thus, there are 2^{12}= 4096 elements in S. To have exactly 5 consecutive 'M's, the following pattern must be present in each trial: {B, M, M, M, M, M, B}

The question then becomes how many elements in the sample space contain this pattern as a subset. This subset can appear as follows:
{B, M, M, M, M, M, B, X, X, X, X, X}
{X, B, M, M, M, M, M, B, X, X, X, X}
{X, X, B, M, M, M, M, M, B, X, X, X}
{X, X, X, B, M, M, M, M, M, B, X, X}
{X, X, X, X, B, M, M, M, M, M, B, X}
{X, X, X, X, X, B, M, M, M, M, M, B}

where X can be either a M or B. Each possibility contains four X's, and the number of elements for each possibility is 2^5=32. So we have 6 × 32 = 192 outcomes.
(10 points)

In addition, we must also take into account the fact that the five M's can appear at the beginning and end of the history as well:
{M, M, M, M, M, B, X, X, X, X, X, X}, {X, X, X, X, X, X, B, M, M, M, M, M}

There are 2×2^6=128 outcomes that match this pattern. But we are counting the following patterns twice: {M, M, M, M, M, B, M, M, M, M, M, B}, {B, M, M, M, M, M, B, M, M, M, M, M}, {M, M, M, M, M, B, B, M, M, M, M, M,}.
(5 points)

So, there are 192+128-3 different elements in S in which the desired subset appears, resulting in a probability of 317/4096.
(5 points)

Problem 3 (15 points)

a) There are three resistors that can fail or operate normally. So the sample space S consists of elements in the form (R_1, R_2, R_3) where R_i represents the status of the resistor *i* in binary: 0 is 0 if the resistor is failed and 1 if the resistor is operational.

S = {(0, 0, 0), (0, 0, 1), (0, 1, 0), (0, 1, 1), (1, 0, 0), (1, 0, 1), (1, 1, 0), (1, 1, 1)}

$|S| = 2^3 = 8$

b) For the current to flow through the circuit:

1. R_3 should not fail and
2. either R_1 or R_2 or both should be operational

This corresponds to a subset of S represented by event A as follows:
A = {(0, 1, 1), (1, 0, 1), (1, 1, 1)}

c) If a resistor is equally likely to be operational or failed, then each element in S has an equal chance of occurring. So P(A) = 3/8. But more generally, if a resistor fails with probability of *p*, it would be operational with probability *p-1* and this wall cause the elements in event A to occur with different probabilities: For example: P{(0, 1, 1)} = p(1-p)(1-p). So we have:

$$P(A) = p(1-p)(1-p) + (1-p)p(1-p) + (1-p)(1-p)(1-p) = p^3 – p^2 – p - 1$$

Problem 4 (20 points) – Sample space of the problem consists of the number of ways that you can choose 5 fruits from 7 bananas,10 grapes, 8 apples, 4 peaches and 5 oranges:

$$Sample\ space\ size = \binom{7+10+8+4+5}{5} = \binom{34}{5}$$

a) **(4 points)**

If you pick one from each group of 7, 10, 8, 4 and 5 fruits: $P = \frac{\binom{7}{1}\times\binom{10}{1}\times\binom{8}{1}\times\binom{4}{1}\times\binom{5}{1}}{\binom{34}{5}}$

b) **(3 points)**

If you pick 3 apples of 8 apples and then another 2 peaches of the 4 peaches available:

$$P = \frac{\binom{8}{3}\times\binom{4}{2}}{\binom{34}{5}}$$

c) **(7 points)**

The possibility can be written as:

1 – (P(no apples) + P(no oranges) - P(no apples, no orange))

$$P = 1 - \frac{\binom{26}{5}+\binom{29}{5}-\binom{21}{5}}{\binom{34}{5}}$$

d) **(6 points)**

There are 3 possibility – zero bananas, one banana, 2 bananas

$$P = \frac{\binom{27}{5} + \binom{7}{1}\times\binom{27}{4} + \binom{7}{2}\times\binom{27}{3}}{\binom{34}{5}}$$

Problem 5(15 points)- An experiment consists of observing the contents of an 8-bit register. We assume that all 256 byte values are equally likely to be observed.

a) (3 points) Any arbitrary 7-digit binary number concatenated with a ZERO is a member of event A. Thus, there are $2^7 = 128$ such numbers out of 256 8-digit numbers. Since all numbers are equally probable, we have:

$$P\{A\} = 128 / 256 = 0.5$$

b) (3 points) $P\{B\} = \binom{8}{6}(\frac{1}{2})^6(\frac{1}{2})^2 = 28 \times 2^{-8}$. The first term indicates the number of different combinations that satisfy the condition that 6 out of 8 digits are ZEROs. The second term indicates the probability that these 6 digits are ZEROs. The third term indicates the probability that the remaining 2 digits are ONEs.

$$P\{B\} = \binom{8}{6}\left(\frac{1}{2}\right)^6\left(\frac{1}{2}\right)^2 = 28 \times 2^{-8} = 0.11$$

c) (3 points) The intersection of event A and B can be described as: the last digit must be a ZERO, and out of the first 7 digits, there are 2 ONEs and 5 ZEROs. Thus, the probability of the intersection is:

$$P\{A \cap B\} = \left(\frac{1}{2}\right)\binom{7}{5}\left(\frac{1}{2}\right)^5\left(\frac{1}{2}\right)^2 = 21 \times 2^{-8} = 0.08$$

Calculating the probability of at least A or B occurring is then straight forward:

$$P\{A \cup B\} = P\{A\} + P\{B\} - P\{A \cap B\} = (128+28-21) \times (2^{-8}) = 0.53$$

d) (3 points) The probability that exactly one of A and B occur can be calculated by:

$$P(A \oplus B) = P\{A \cup B\} - P\{A \cap B\} = P\{A\} + P\{B\} - 2P\{A \cap B\} = (128+28-42) \times (2^{-8}) = 0.45$$

e) (3 points) What is the probability of at least one of A or B does not occur?

$$P\{A^c \cup B^c\} = 1 - P\{A \cap B\} = 1 - (21 \times 2^{-8}) = 0.92$$

Problem 6 (10 points)–

a) **(2 points)**

There are two players in the game, and each player has 2 possible actions: confess (C) or deny (same as remaining silent) (D). So the sample of space of the problem consists of:

$$S = \{(C, C), (C, D), (D, C), (D, D)\}$$

b) **(4 points)**

0.6, because A is confessing and B won't confess with a probability of 0.6.

c) **(4 points)**

0.4, because A is confessing and B will confess with a probability of 0.4.