ECE 313
Homework 12 Solution

Problem 1 –

a) Find the value of c.

$$\int_{-4}^{4} c = 1 \;\Rightarrow\; c = \frac{1}{8}$$

b) Determine the ML decision rule.

Decide H_0 if $|X| > 1.5$ and H_1 otherwise.

c) Determine the false alarm and miss detection probabilities for the ML decision rule.

False alarm probability = $3 * \frac{1}{8} = \frac{3}{8}$
Miss detection = $2 * \frac{1}{2} * \frac{1}{2} * \frac{1}{8} = \frac{1}{16}$

d) Determine the MAP decision rule.

Decide H_0 for all X (never select H_1)

e) Determine the false alarm and miss detection probabilities for the MAP decision rule.

Miss detection probability = 1
False alarm probability = 0

f) Determine the error probability for both the ML and MAP decision rules.

ML error = $0.2 * \frac{1}{16} + 0.8 * \frac{3}{8}$
MAP error = $0.2 * 1$

Problem 2 –

a) Compute the likelihood ratio $\Lambda(Y)$.

$$\Lambda(y) = \frac{1}{3}\exp\left(|y| - \frac{|y-3|}{3}\right)$$

b) What are the values of π_0 and π_1 for this problem?

$\pi_0 = \frac{3}{4}$ and. $\pi_1 = \frac{1}{4}$

c) Determine the MAP decision rule to identify the type of an orange from its weight. Express the decision rule in a simplified form.

If $y < 0$, then the decision rule is select H_0 if $y > -\alpha$
If $0 < y < 3$, then the decision rule is select H_0 if $y < \frac{\alpha}{2}$
If $y > 3$, then the decision rule if select H_0 if $y < \alpha - 3$
Where $\alpha = \frac{3 \ln 9 + 3}{2}$
Combining all of this information togther, we find that the decision rule is select H_0 if $-\alpha < y < \frac{\alpha}{2}$ and select H_1 otherwise.

d) Under what circumstances would the MAP and ML decision rules be the same for this problem?

If $\pi_0 = \pi_1 = \frac{1}{2}$

Problem 3 – The ML decision rule is defined by underlying the larger probability in each column in the likelihood matrix:

	X = 0	X = 1	X = 2	X = 3	X = 4
H_1	0.00	0.10	0.24	0.34	0.32
H_0	0.08	0.15	0.31	0.31	0.15

So the ML rule declares H1for X ≥ 3, and H0 otherwise.
The conditional probability of false alarm can be calculated by adding the elements in the row H1 of the likelihood matrix that are not underlined by the decision rule:
$p_{ML-false-alarm} = 0.31 + 0.15 = 0.46$

The conditional probability of false alarm can be calculated by adding the elements in the row H0 of the likelihood matrix that are not underlined by the decision rule:
$p_{ML-miss} = 0.00 + 0.10 + 0.24 = 0.34$

Probability of error can be calculated as follows:
$p_{ML-e} = \pi_0 p_{ML-false-alarm} + \pi_1 p_{ML-miss} = (0.3)(0.46) + (0.7)(0.34) = 0.376$

The MAP decision rule is defined by underlying the larger probability in each column of the joint probability matrix. We calculate the joint probability matrix by multiplying $\pi_1 = 0.7$ *and* $\pi_0 = 0.3$ to the rows H1 and H0 of the likelihood matrix, respectively,:

	X = 0	X = 1	X = 2	X = 3	X = 4
H_1	0.000	0.07	0.168	0.238	0.224
H_0	0.024	0.045	0.093	0.093	0.045

So the MAP rule declares H1for X ≥ 2, and H0 otherwise. For the error

$p_{MAP-false-alarm} = 0.31 + 0.31 + 0.15 = 0.77$
$p_{MAP-miss} = 0.00 + 0.10 = 0.10$
$p_{MAP-e} = \pi_0 p_{MAP-false-alarm} + \pi_1 p_{MAP-miss} = (0.3)(0.77) + (0.7)(0.1) = 0.30$

Problem 4 – $\frac{\pi_0}{\pi_1} = 2$ and we know that $\pi_0 + \pi_1 = 1$. So: $\pi_0 = \frac{2}{3}$ *and* $\pi_1 = \frac{1}{3}$

a) When no attacks are happening (*H₀* is true), the number of requests, *X*, arriving at the Web Server from one client in an interval of 1 second is Poisson distributed with parameter $\lambda = 3$. So the conditional pmf of X given H_0 can be written as follows:

$$P(X = k|H_0) = \frac{e^{-3}3^{k}}{k!}$$

b) When an attack is occurring (H_1 is true), *X* is a binomial random variable with n =100 and p = 0.04, so the conditional pmf of X given H_0 can be written as follows:

$$P(X = k|H_1) = \binom{100}{k}(0.04)^k(0.06)^{100-k}$$

The likelihood ratio test can be written as follows:

$$\Lambda(X) = \frac{P(X = k|H_1)}{P(X = k|H_0)} = \frac{\frac{e^{-3}3^{k}}{k!}}{\binom{100}{k}(0.04)^k(0.06)^{100-k}}$$

c) For the ML Rule the threshold of the likelihood ratio test is τ = 1:
So H_1 is declared when $\Lambda(X) > 1$ which means $k \geq 4$ **(See the following table)**

d) MAP Rule the threshold of the likelihood ratio test is $\tau = \frac{\pi_0}{\pi_1} = 2$:
So H_1 is declared when $\Lambda(X) > 2$ which means $k \geq 6$ **(See the following table)**

X	ML		MAP	
	P(X = k|H1)	P(X = k|H1)	P(X =k |H1)	2*P(X = k | H0)
0	0.01687	0.05	0.01687	0.03374
1	0.07029	0.149	0.07029	0.14058
2	0.14498	0.224	0.14498	0.28996
3	0.19733	0.224	0.19733	0.39466
4	0.19939	0.168	0.19939	0.39878
5	0.15951	0.101	0.15951	0.31902
6	0.10523	0.05	0.10523	0.21046
7	0.05888	0.022	0.05888	0.11776
8	0.02852	0.008	0.02852	0.05704
9	0.01215	0.003	0.01215	0.0243
10	0.00461	0.001	0.00461	0.00922

11	0.00157	0	0.00157	0.00314
12	0.00049	0	0.00049	0.00098

e) For the ML Rule we have:

$$p_{ML-false-alarm} = P(H_1\ declared\ |H_0\ true) = P(X \geq 4\ |H_0)$$

$$= 1 - P(X < 4 \mid H_0) = 1 - \sum_{0}^{3} \frac{e^{-3}3^{k}}{k!} = 0.36$$

$$p_{ML-miss} = P(H_0\ declared\ |H_1\ true) = P(X < 4\ |H_1)$$

$$= \sum_{0}^{3} \binom{100}{k} (0.04)^{k}(0.06)^{100-k} = 0.43$$

$$p_{ML-e} = \pi_0 p_{ML-false-alarm} + \pi_1 p_{ML-miss} = (0.67)(0.36) + (0.33)(0.07) = 0.26$$

f) For the MAP Rule we have:

$$p_{MAP-false-alarm} = P(H_1\ declared\ |H_0\ true) = P(X \geq 6\ |H_0)$$

$$= 1 - P(X < 6 \mid H_0) = 1 - \sum_{0}^{5} \frac{e^{-3}3^{k}}{k!} = 0.08$$

$$p_{MAP-miss} = P(H_0\ declared\ |H_1\ true) = P(X < 6\ |H_1)$$

$$= \sum_{0}^{5} \binom{100}{k} (0.04)^{k}(0.06)^{100-k} = 0.79$$

$$p_{MAP-e} = \pi_0 p_{MAP-false-alarm} + \pi_1 p_{MAP-miss} = (0.67)(0.08) + (0.33)(0.79) = 0.31$$