ECE 313
Homework 11
Due Date: Wednesday, May 15, 2024

Write your name and NetID on top of all the pages. **Show your work to get partial credit.**

Markov Inequality

Problem 1 –

Solution: First using Markov inequality, we have,

(a)

$$P(X > 5) \le \frac{E[X]}{5} = \frac{4.39}{5} = 0.876.$$

(b)

$$P(X > 1) \le \frac{E[X]}{1} = 4.39 \text{ (not very useful!)}.$$

Using the distributional information,

(a)

$$P(X > 5) = e^{-\frac{5}{4.39}} = e^{-1.13895} \approx 0.32.$$

(b)

$$P(X > 1) = e^{-\frac{1}{4.39}} \approx 0.7963.$$

Thus the bounds provided by the Markov inequality are not very sharp in this case.

<u>Law of Large Numbers</u>

Problem 2 –

Let X be the number of heads in each trial of tossing 3 fair coins.

a) X has a binomial distribution. P(X) is as follows:

X	P(X)
0	$\left(\frac{1}{2}\right)^3 = \frac{1}{8}$
1	$3 \times \left(\frac{1}{2}\right)^1 \times \left(\frac{1}{2}\right)^2 = \frac{3}{8}$
2	$\frac{3 \times 2}{2} \times \left(\frac{1}{2}\right)^2 \times \left(\frac{1}{2}\right)^1 = \frac{3}{8}$
3	$\left(\frac{1}{2}\right)^3 = \frac{1}{8}$

$$\mathrm{E[X]} = \sum_{x=0}^{3} xp(x) = 0 \times \frac{1}{8} + 1 \times \frac{3}{8} + 2 \times \frac{3}{8} + 3 \times \frac{1}{8} = 1.5$$

$$\mathrm{E[X^2]} = \sum_{x=0}^{3} x^2p(x) = 0 \times \frac{1}{8} + 1 \times \frac{3}{8} + 4 \times \frac{3}{8} + 9 \times \frac{1}{8} = 3$$

$$\mathrm{Var[X]} = \mathrm{E[X^2]} - \mathrm{E[X]}^2 = 3 - 1.5^2 = 0.75$$

$$\sigma_X = \sqrt{Var(X)} = 0.867$$

b) So mean $\mu = 1.5$ and variance $\sigma^2 = 0.75$. If we let *S* the sum of numbers showing up in 1000 independent tosses of 3 coins ($S = X_1 + X_2 + X_3 + \ldots + X_{1000}$), by the law of large numbers, $\frac{S}{n} = \frac{X_1+X_2+\cdots X_n}{n} \to \mu \quad as\ n \to \infty$, so we expect the sum S to be near 1500.

Central Limit Theorem

Problem 3 – If we let X_i denote the lifetime of the ith battery to be put in use, then we desire $p = P\{X_1 + \cdots + X_{25} > 1100\}$, which is approximated as follows:

$$p = P\left\{\frac{X_1 + \cdots + X_{25} - 1000}{20\sqrt{25}} > \frac{1100 - 1000}{20\sqrt{25}}\right\}$$

$$\approx P\{N(0,1) > 1\}$$

$$= 1 - \Phi(1)$$

$$\approx 0.1587$$

Achieving potential in a class

Problem 4 –

a) Let X_i denotes the score of the i[th] part and $S_{100} = X_1 + X_2 + ... + X_{100}$ denotes the final score. Since E[X_i] = 0.9*5+0.1*0 = 4.5, the expected value of S_{100} is (100)(4.5) = **450**.

b) $P\{S_{100} \geq 425\} = Q\left(\frac{425-450}{\sqrt{(5^2*0.9-4.5^2)*100}}\right) = \mathbf{Q\left(-\frac{5}{3}\right) \approx 0.9522}$

c) $\text{Var}(X_i) = \text{E}[X_i{}^2] - E[X_i]^2 = 2.25$

$\text{Cov}(X_i, X_j) = \sqrt{Var(X_i)Var(X_j)}\rho_{X_i, X_j} = 1.8$

$\text{E}[Y_1] = \text{E}[X_1 + X_2 + X_3 + X_4] = 4 * 4.5 = \mathbf{18}$

$Var(Y_1) = \text{Var}(X_1 + X_2 + X_3 + X_4) = 4 * 2.25 + 2 * 6 * 1.8 = \mathbf{30.6}$

d) $P\{S \geq 425\} = Q\left(\frac{425-450}{5\sqrt{30.6}}\right) \approx \mathbf{Q(-0.9039) = 0.81697}$