ECE 313
Homework 10 Solution

Problem 1 –

Answer:

$\boxed{N \text{ is independent of } X_N}$.

The distribution of X_N is $X_N \sim \text{Uniform}(c, 1)$. And knowing the the X_N will not change this distribution, and thus, will not change the distribution of N.

Problem 2 –

Answer:

Let $Z = \frac{X}{Y}$.

So, the CDF of Z is ($\frac{X}{Y} \leq Z$ *which is* $Y \geq \frac{X}{Z}$):

$$F_Z(z) = \int_0^\infty \int_{\frac{x}{z}}^\infty e^{-(x+y)} dy dx$$
$$= \int_0^\infty e^{-\frac{z+1}{z}x} dx$$
$$= \frac{z}{z+1}$$

So, the density function of Z is:

$$f_Z(z) = F'_Z(z) = (\frac{z}{z+1})' = \frac{1}{(z+1)^2}$$

That is,

$$\boxed{f_{\frac{X}{Y}}(x,y) = \frac{1}{(\frac{x}{y}+1)^2}}$$

Problem 3 –

a)

$$f_X(x) = \int_0^\infty e^{-\frac{x}{\alpha}} y e^{-y^2} dy = e^{-\frac{x}{\alpha}} \int_0^\infty y e^{-y^2} dy = e^{-\frac{x}{\alpha}} \left[-\frac{1}{2} e^{-y^2}\right]_0^\infty = \frac{1}{2} e^{-\frac{x}{\alpha}}$$

$$f_Y(x) = \int_0^\infty e^{-\frac{x}{\alpha}} y e^{-y^2} dx = y e^{-y^2} \int_0^\infty e^{-\frac{x}{\alpha}} dx = y e^{-y^2} \left[-\alpha e^{-\frac{x}{\alpha}}\right]_0^\infty = \alpha y e^{-y}$$

b) For X and Y to be independent, $f_{X,Y}(x,y) = f_X(x) f_Y(y) = \frac{1}{2} \alpha e^{-\frac{x}{\alpha}} y e^{-y}$

Therefore, $\alpha = 2$

Problem 4 –

a) We have: Var(X + 2Y) = Var(X) + Var(2Y) + 2Cov(X, 2Y)
Var(X− 2Y) = Var(X) + Var(2Y) − 2Cov(X, 2Y)

From the expressions for Var(X + 2Y) and Var(X − 2Y) in terms of the variances and covariance, the condition Var(X + 2Y) = Var(X − 2Y) implies that Cov(X, Y) = 0. Hence, X and Y are uncorrelated.

But we cannot say if they are independent or not.
If X and Y are independent then Cov(X, Y) = 0, but the reverse might not be true.

b) No. The condition Var(X) = Var(Y) does not imply that Cov(X, Y) = 0.

Problem 5 –

$$Cov(X_i, Y_j) = \begin{cases} \frac{3}{4} * 4 = 3, & if\ i = j \\ -\frac{1}{4} * 4 = -1, & if\ |i-j| = 1 \\ 0, & else. \end{cases}$$

$$Cov(W, Z) = \sum_{i=j;1\le i\le n} Cov(X_i, Y_j) + \sum_{|i-j|=1;1\le i,j\le n} Cov(X_i, Y_j) + 0$$

$$= 3n + \sum_i Cov(X_i, Y_{i-1}) + \sum_i Cov(X_i, Y_{i+1}) = 3n - 2(n-1) = n + 2$$