ECE 313
Homework 10
Due Date: Wednesday, May 1, 2024

Write your name and NetID on top of all the pages. **Show your work to get partial credit.**

Problem 1

Let $X_1, X_2, \ldots$ be a sequence of independent Uniform(0, 1) random variables. For a fixed constant c, define the random variable N by

$$N = \min\{n : X_n > c\}$$

Is N independent of X_N? That is, does knowing the value of the first random variable that is greater than c affect the probability distribution of when this random variable occurs?

Problem 2

The joint density of X and Y is given by

$$f(x,y) = \begin{cases} e^{-(x+y)} & 0 < x < \infty, 0 < y < \infty \\ 0 & \text{otherwise} \end{cases}$$

Find the density function of the random variable X/Y.

Problem 3 – Let the joint pdf of X and Y be given by:

$$f(x,y) = e^{-\frac{x}{\alpha}} y e^{-y^2}, for\ x > 0, y > 0$$

where $\alpha \neq 0$. The random variables X and Y are said to have a two-dimensional (or bivariate) normal pdf.

a) Show that the marginal pdf's of X and Y are:

$$f(x) = \frac{1}{2} e^{-\frac{x}{\alpha}} \qquad and \qquad f(y) = \alpha y e^{-y^2}$$

b) Find the values of α, for which X and Y are independent.

Problem 4 – Consider two random variables X and Y:

a) If $\mathrm{Var}(X + 2Y) = \mathrm{Var}(X - 2Y)$, are X and Y uncorrelated? Are they independent? Why?

b) If $\mathrm{Var}(X) = \mathrm{Var}(Y)$, are X and Y uncorrelated? Why?

Problem 5

[The covariance of sums of correlated random variables]
Suppose $X_1, \ldots X_n$ and $Y_1, \ldots, Y_n$ are random variables on a common probability space such that $\mathrm{Var}(X_i) = \mathrm{Var}(Y_i) = 4$ for all i, and

$$\rho_{X_i,Y_j} = \begin{cases} 3/4 & \text{if } i = j \\ -1/4 & \text{if } |i-j| = 1 \\ 0 & \text{else.} \end{cases}$$

Let $W = \sum_{i=1}^{n} X_i$ and $Z = \sum_{i=1}^{n} Y_i$. Express $\mathrm{Cov}(W, Z)$ as a function of n.