ZJU-UIUC INSTITUTE

Final Examination

(For Students, please fill in your name and ID number and read any instructions below before starting your exam. Please be aware of your obligation not to receive or give aid to others. Don't take the test out of the exam room.)

Name:	ID:

(For instructors, please complete the form below)

Course: ECE 313	Semester: Fall 2021	Instructor: Prof. Butala	
Exam Type: Closed-book ■ Open-book ☐ Partly Open-book ☐ Take Home ☐			
Exam Date: 2021.12.29	Start Time: 9:00	End Time: 12:00	Duration: 3 hours
Total number of pages: 12		Number of questions: 8	
Specific requirements and instructions to students: When the exam begins, please read the Instructions on page 2.			

(For marker, please fill in the score and grade and sign below.)

Score:	Grade:
Signature:	Date:

(Please go on to the next page for questions)

ZJU-UIUC Institute **Prof. Butala**
ECE 313, Fall 2021

Final Examination

9:00 – 12:00, Wednesday, December 29, 2021

Name: ____________________________

ID Number: ______________________

Question	Points	Score
1	24	
2	9	
3	15	
4	10	
5	10	
6	12	
7	8	
8	12	
Total:	100	

Instructions

- You may not use any books, calculators, or notes other than three two-sided sheets of A4 paper.
- You may not use a calculator, cell phone, or other electronic devices during the exam.
- When you are asked to "calculate," "determine," or "find," this means providing closed-form expressions (i.e., without summation or integration signs).
- Neatness counts. If we are unable to read your work, we cannot grade it.
- Turn in your entire booklet once you are finished. No extra papers will be considered.
- You are NOT required to simplify your solutions. They must be in closed form (e.g., no summations or integrals) unless stated otherwise. For example, you do not have to simplify or reduce fractions, expand (n choose k) operations, etc.
- Show your work / process. A correct final answer does not guarantee full credit and an incorrect final answer does not necessarily mean you will lose credit.

1. (24 points) Select either True or False for each statement below and justify your response (an answer without justification will receive no credit — your justification does not have to be elaborate, a one sentence statement should be sufficient in most cases). In order to discourage guessing, 1 point will be deducted for each incorrect answer and/or justification (no penalty or gain for blank answers). A net negative score will reduce your total exam score.

   (a) (2 points) Consider the random variables X, Y, and Z. If X and Y are independent, Y and Z are independent, and X and Z are independent, then X, Y, and Z are mutually independent. ○ True ○ False

   (b) (2 points) Consider the random variables X, Y, and Z. If X, Y, and Z are mutually independent then X and Y are independent, Y and Z are independent, and X and Z are independent. ○ True ○ False

   (c) (2 points) Consider the discrete-type random variables X and Y. If they are independent then $P\{X = a, Y = b\} = P\{X = a\} P\{Y = b\}$ for all $a \in \mathbb{R}$ and $b \in \mathbb{R}$. ○ True ○ False

   (d) (2 points) If X and Y are continuous-type random variables, then $P\{X = a, Y = b\} = P\{X = a\} P\{Y = b\}$ for all $a \in \mathbb{R}$ and $b \in \mathbb{R}$. ○ True ○ False

(e) (2 points) Consider the random variables X and Y. If X and Y are uncorrelated then $\mathrm{E}[XY] = 0$. ◯ True ◯ False

(f) (2 points) If $\mathrm{Cov}(X, Y) = 0$ and the random variables (X, Y) are jointly Gaussian, then X and Y are independent. ◯ True ◯ False

(g) (2 points) A triple modular redundant (TMR) system is always more reliable than a simplex system. ◯ True ◯ False

(h) (2 points) The central limit theorem only applies to normal random variables. ◯ True ◯ False

(i) (2 points) A ML decision rule to decide between hypotheses H_0 and H_1 can always be written in the form $f(x) = \begin{cases} H_0 & x > \alpha \\ H_1 & \text{otherwise} \end{cases}$ where x is the observation and α is a constant. ◯ True ◯ False

(j) (2 points) A MAP decision rule to decide between hypotheses H_0 and H_1 can always be written in the form $g(x) = \begin{cases} H_0 & x > \beta \\ H_1 & \text{otherwise} \end{cases}$ where x is the observation and β is a constant. ◯ True ◯ False

(k) (2 points) ML and MAP decision rules can be equivalent. ◯ True ◯ False

(l) (2 points) The pdf of the random variable X is a symmetric function, i.e., there exists a value x_0 such that $f_X(x_0+\delta) = f_X(x_0-\delta)$ for all $\delta \in \mathbb{R}$. We can always conclude that the median of X is equal to x_0. ◯ True ◯ False

2. (9 points) The campus has a contract with two vendors: one has red beverage vending machines and the other vendor has blue machines. Unfortunately, the machines are prone to error and one type will dispense a beverage successfully only 70% of the time and the other type is far worse, dispensing successfully only 40% of the time. You cannot recall which vendor has the more reliable vending machines.

   You leave your dorm room to purchase a beverage. You decide to use a red vending machine by flipping a fair coin.

   (a) (3 points) What is the probability that the red machine will have an error and not successfully dispense your beverage after a single attempt?

   (b) (3 points) Suppose you were successful on your third attempt. What is the probability that the red machine is the less reliable type?

   (c) (3 points) What is the expected number of attempts required before the red machine will dispense a beverage?

3. (15 points) Alice, Bob, and Chris are roommates and share an apartment. They decide on the following game to determine who has to clean the apartment this week. Each tosses a fair coin. If two of the coins show the same face and the third coin shows a different face, the tosser of the third coin cleans the apartment. Otherwise, it must be that all three coins are showing heads or they all are showing tails, in which case another round of coin tossing occurs.

   (a) (3 points) What is the probability that at least three rounds of tosses are required to determine who cleans?

   (b) (3 points) What is the probability that the decision of who cleans is made on an even-numbered round?

(c) (3 points) What is the probability that at least three rounds were required given that the decision of who cleans was made on an even numbered round?

(d) (3 points) Chris thinks it is unfair that he cleaned the apartment every week last month, and decides to attempt to improve his chances by secretly replacing his fair coin with a two-headed coin. The others continue to toss fair coins. What is the probability that Chris has to clean the apartment this week?

(e) (3 points) Bob also decides to secretly replace his fair coin with a two-tailed coin. Note that now only Alice is tossing a fair coin. What is the probability that Chris has to clean the apartment this week?

4. (10 points) The lifetime of batteries are independent exponential random variables, each with parameter λ. A flashlight needs two batteries to function. Suppose you have a flashlight and a stockpile of n batteries. Let T be the total time that that flashlight can operate. What is the distribution of T?

5. (10 points) Consider the following cooling system for a server rack composed of two subsystems S_1 and S_2. Subsystem S_1 is composed of two coolers A and B, and subsystem S_1 fails if either A or B fail. Subsystem S_2 has a single cooler C which acts as a backup of subsystem S_1 and will be powered up only if subsystem S_1 fails. The overall system fails if both S_1 and S_2 fail.

   Assume failure detection and subsystem switching is perfect. Model the lifetimes of the coolers A, B, and C with three independent random variables $X_1 \sim \text{Exponential}(\lambda)$, $X_2 \sim \text{Exponential}(\lambda)$, and $X_3 \sim \text{Exponential}(3\lambda)$, respectively.

   (a) (5 points) What is the lifetime of the overall system?

   (b) (5 points) What is the failure rate function of the overall system?

6. (12 points) Suppose X and Y are zero-mean, unit-variance jointly Gaussian random variables with correlation coefficient $\rho = 0.8 = \frac{4}{5}$.

   (a) (4 points) Determine $\mathrm{Var}(5X - 3Y)$.

   (b) (4 points) Determine $\mathrm{P}\{(5X - 3Y)^2 \geq 40\}$.

   (c) (4 points) Determine $\mathrm{E}[Y \mid X = 2]$.

7. (8 points) Let $Y \sim \text{Uniform}(-1, 1)$ and $X = Y^2$.

   (a) (2 points) What is the MMSE estimator of X given $Y = v$?

   (b) (2 points) What is the MMSE estimator of Y given $X = u$?

   (c) (2 points) What is the LMMSE estimator of X given $Y = v$?

   (d) (2 points) What is the LMMSE estimator of Y given $X = u$?

8. (12 points) Consider the game of darts. Suppose each thrown dart lands a distance X_i from the target center where X_i is a random variable with the Rayleigh distribution, i.e., the pdf of X_i is

$$f_{X_i}(x_i) = \begin{cases} \frac{x_i}{\theta} e^{-\frac{x_i^2}{2\theta}} & x_i \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

and $\theta > 0$ is the distribution scale parameter.

(a) (6 points) If dart throws are independent and you measure the result of n trials, i.e., $X_1 = x_1, \ldots, X_n = x_n$, what is the maximum likelihood estimate for the scale parameter θ?

(b) (6 points) Suppose the scale parameter for a professional player is equal to θ_p and is equal to $\theta_a = 10\theta_p$ for an amateur player. After n independent dart throws with $X_1 = x_1, \ldots, X_n = x_n$, what is the maximum likelihood decision rule to determine if a player is a professional or an amateur? Simplify your solution.