ZJU-UIUC Institute **Prof. Butala**
ECE 313, Fall 2021

Final Examination

9:00 – 12:00, Wednesday, December 29, 2021

Name: ______________________

ID Number: ______________________

Question	Points	Score
1	24	
2	9	
3	15	
4	10	
5	10	
6	12	
7	8	
8	12	
Total:	100	

Instructions

- You may not use any books, calculators, or notes other than three two-sided sheets of A4 paper.
- You may not use a calculator, cell phone, or other electronic devices during the exam.
- When you are asked to "calculate," "determine," or "find," this means providing closed-form expressions (i.e., without summation or integration signs).
- Neatness counts. If we are unable to read your work, we cannot grade it.
- Turn in your entire booklet once you are finished. No extra papers will be considered.
- You are NOT required to simplify your solutions. They must be in closed form (e.g., no summations or integrals) unless stated otherwise. For example, you do not have to simplify or reduce fractions, expand (n choose k) operations, etc.
- Show your work / process. A correct final answer does not guarantee full credit and an incorrect final answer does not necessarily mean you will lose credit.

1. (24 points) Select either True or False for each statement below and justify your response (an answer without justification will receive no credit — your justification does not have to be elaborate, a one sentence statement should be sufficient in most cases). In order to discourage guessing, 1 point will be deducted for each incorrect answer and/or justification (no penalty or gain for blank answers). A net negative score will reduce your total exam score.

(a) (2 points) Consider the random variables X, Y, and Z. If X and Y are independent, Y and Z are independent, and X and Z are independent, then X, Y, and Z are mutually independent. ◯ True ◯ False

Solution: False. Pairwise Independence does not imply mutual independence.

(b) (2 points) Consider the random variables X, Y, and Z. If X, Y, and Z are mutually independent then X and Y are independent, Y and Z are independent, and X and Z are independent. ◯ True ◯ False

Solution: True. Mutual independence does imply pairwise independence.

(c) (2 points) Consider the discrete-type random variables X and Y. If they are independent then $\mathrm{P}\{X = a, Y = b\} = \mathrm{P}\{X = a\}\,\mathrm{P}\{Y = b\}$ for all $a \in \mathbb{R}$ and $b \in \mathbb{R}$. ◯ True ◯ False

Solution: True. This is the definition of what it means for X and Y to be independent.

(d) (2 points) If X and Y are continuous-type random variables, then $\mathrm{P}\{X = a, Y = b\} = \mathrm{P}\{X = a\}\,\mathrm{P}\{Y = b\}$ for all $a \in \mathbb{R}$ and $b \in \mathbb{R}$. ◯ True ◯ False

Solution: True. Since X and Y are continuous-type then $\mathrm{P}\{X = a, Y = b\} = \mathrm{P}\{X = a\} = \mathrm{P}\{Y = b\} = 0$.

(e) (2 points) Consider the random variables X and Y. If X and Y are uncorrelated then $\mathrm{E}[XY] = 0$. ◯ True ◯ False

Solution: False. If X and Y are uncorrelated then $\mathrm{Cov}(X, Y) = \mathrm{E}[XY] - \mathrm{E}[X]\,\mathrm{E}[Y] = 0$ which does not imply that $\mathrm{E}[XY] = 0$.

(f) (2 points) If $\mathrm{Cov}(X, Y) = 0$ and the random variables (X, Y) are jointly Gaussian, then X and Y are independent. ◯ True ◯ False

Solution: True. If X and Y are jointly Gaussian and uncorrelated then they are independent.

(g) (2 points) A triple modular redundant (TMR) system is always more reliable than a simplex system. ◯ True ◯ False

> **Solution:** False. A simplex system has greater reliability than a TMR system if the component reliability is less than $\frac{1}{2}$.

(h) (2 points) The central limit theorem only applies to normal random variables. ◯ True ◯ False

> **Solution:** False. The central limit theorem applies to independent, identically distributed random variables.

(i) (2 points) A ML decision rule to decide between hypotheses H_0 and H_1 can always be written in the form $f(x) = \begin{cases} H_0 & x > \alpha \\ H_1 & \text{otherwise} \end{cases}$ where x is the observation and α is a constant. ◯ True ◯ False

> **Solution:** True. A ML decision rule can always be written as a likelihood ratio test with $\alpha = 1$.

(j) (2 points) A MAP decision rule to decide between hypotheses H_0 and H_1 can always be written in the form $g(x) = \begin{cases} H_0 & x > \beta \\ H_1 & \text{otherwise} \end{cases}$ where x is the observation and β is a constant. ◯ True ◯ False

> **Solution:** True. A MAP decision rule can always be written as a likelihood ratio test with $\alpha = \frac{\pi_0}{\pi_1}$.

(k) (2 points) ML and MAP decision rules can be equivalent. ◯ True ◯ False

> **Solution:** True. The ML and MAP decision rules are equal when $\pi_0 = \pi_1 = \frac{1}{2}$.

(l) (2 points) The pdf of the random variable X is a symmetric function, i.e., there exists a value x_0 such that $f_X(x_0 + \delta) = f_X(x_0 - \delta)$ for all $\delta \in \mathbb{R}$. We can always conclude that the median of X is equal to x_0. ◯ True ◯ False

> **Solution:** The answer is True.

2. (9 points) The campus has a contract with two vendors: one has red beverage vending machines and the other vendor has blue machines. Unfortunately, the machines are prone to error and one type will dispense a beverage successfully only 70% of the time and the other type is far worse, dispensing successfully only 40% of the time. You cannot recall which vendor has the more reliable vending machines.

   You leave your dorm room to purchase a beverage. You decide to use a red vending machine by flipping a fair coin.

(a) (3 points) What is the probability that the red machine will have an error and not successfully dispense your beverage after a single attempt?

Solution: Let V be the event that the machine successfully dispenses the beverage on the first attempt and let R be the event that the red machine is the more reliable type. Then

$$\begin{aligned} P\{V\} &= P\{V \mid R\}\, P\{R\} + P\{V \mid R^c\}\, P\{R^c\} \\ &= \left(\frac{7}{10}\right)\left(\frac{1}{2}\right) + \left(\frac{4}{10}\right)\left(\frac{1}{2}\right) = \frac{11}{20} \end{aligned}$$

(b) (3 points) Suppose you were successful on your third attempt. What is the probability that the red machine is the less reliable type?

Solution: Let V_3 be the event that the machine successfully dispenses the beverage on the third attempt:

$$\begin{aligned} P\{V_3\} &= P\{V_3 \mid R\}\, P\{R\} + P\{V_3 \mid R^c\}\, P\{R^c\} \\ &= \left(\frac{3}{10}\right)\left(\frac{3}{10}\right)\left(\frac{7}{10}\right)\left(\frac{1}{2}\right) + \left(\frac{6}{10}\right)\left(\frac{6}{10}\right)\left(\frac{4}{10}\right)\left(\frac{1}{2}\right) \\ &= \frac{42+144}{2000} = \frac{186}{2000} = \frac{93}{1000} \end{aligned}$$

Then

$$\begin{aligned} P\{R^c \mid V_3\} &= \frac{P\{V_3 \mid R^c\}\, P\{R^c\}}{P\{V_3\}} \\ &= \frac{\left(\frac{6}{10}\right)^2\left(\frac{4}{10}\right)\left(\frac{1}{2}\right)}{\frac{93}{1000}} \\ &= \left(\frac{144}{2000}\right)\left(\frac{2000}{186}\right) = \frac{144}{(2)(186)} = \frac{12}{31} \end{aligned}$$

(c) (3 points) What is the expected number of attempts required before the red machine will dispense a beverage?

Solution: Let A be the number of attempts before the machine successfully dispenses a beverage.

$$\begin{aligned} E[A] &= E[A \mid R]\, P\{R\} + E[A \mid R^c]\, P\{R^c\} \\ &= \left(\frac{10}{7}\right)\left(\frac{1}{2}\right) + \left(\frac{10}{4}\right)\left(\frac{1}{2}\right) = \frac{55}{28} \end{aligned}$$

3. (15 points) Alice, Bob, and Chris are roommates and share an apartment. They

decide on the following game to determine who has to clean the apartment this week. Each tosses a fair coin. If two of the coins show the same face and the third coin shows a different face, the tosser of the third coin cleans the apartment. Otherwise, it must be that all three coins are showing heads or they all are showing tails, in which case another round of coin tossing occurs.

(a) (3 points) What is the probability that at least three rounds of tosses are required to determine who cleans?

Solution:

$$\begin{aligned} P\{\text{HHH} \cup \text{TTT}\} &= \frac{1}{8} + \frac{1}{8} = \frac{1}{4} \\ P\{X = 1\} &= \frac{3}{4} \\ P\{X = 2\} &= \left(\frac{1}{4}\right)\left(\frac{3}{4}\right) = \frac{3}{16} \\ \implies P\{X \geq 3\} &= 1 - \frac{3}{4} - \frac{3}{16} = \frac{16 - 12 - 3}{16} = \frac{1}{16} \end{aligned}$$

(b) (3 points) What is the probability that the decision of who cleans is made on an even-numbered round?

Solution:

$$P\{X = 2 \cup X = 4 \cup X = 6 \cup \dots\}$$

$$\begin{aligned} &= \left(\frac{1}{4}\right)\left(\frac{3}{4}\right) + \left(\frac{1}{4}\right)^3\left(\frac{3}{4}\right) + \left(\frac{1}{4}\right)^5\left(\frac{3}{4}\right) + \dots \\ &= \left(\frac{3}{4}\right)\sum_{i=0}^{\infty}\left(\frac{1}{4}\right)^{2i+1} \\ &= \left(\frac{3}{4}\right)\left(\frac{1}{4}\right)\sum_{i=0}^{\infty}\left(\frac{1}{16}\right) \\ &= \frac{\left(\frac{3}{4}\right)\left(\frac{1}{4}\right)}{1 - \left(\frac{1}{16}\right)} = \left(\frac{3}{16}\right)\left(\frac{16}{15}\right) = \frac{1}{5} \end{aligned}$$

(c) (3 points) What is the probability that at least three rounds were required given that the decision of who cleans was made on an even numbered round?

Solution:

$$\begin{aligned} P\{X \geq 3 \mid X \text{ is even}\} &= \frac{P\{X \geq 3, X \text{ is even}\}}{P\{X \text{ is even}\}} \\ &= \frac{P\{X = 4, X = 6, \dots\}}{P\{X = 2, X = 4, X = 6, \dots\}} \\ &= \frac{\frac{1}{5} - \left(\frac{1}{4}\right)\left(\frac{3}{4}\right)}{\frac{1}{5}} = 1 - \frac{(5)(3)}{16} = \frac{1}{16} \end{aligned}$$

(d) (3 points) Chris thinks it is unfair that he cleaned the apartment every week last month, and decides to attempt to improve his chances by secretly replacing his fair coin with a two-headed coin. The others continue to toss fair coins. What is the probability that Chris has to clean the apartment this week?

Solution: Note the possible outcomes (C=Chris, A=Alice, B=Bob):

C	A	B	Outcome
H	T	T	Chris cleans
H	H	T	Bob cleans
H	T	H	Alice cleans
H	H	H	Flip again

When someones cleans, there is only one outcome with Chris cleaning. There are 3 outcomes when someone cleans and each of these possibilities are equally likely.

$$\implies P\{\text{Chris cleans}\} = \frac{1}{3}$$

(e) (3 points) Bob also decides to secretly replace his fair coin with a two-tailed coin. Note that now only Alice is tossing a fair coin. What is the probability that Chris has to clean the apartment this week?

Solution: Note the possible outcomes:

B	C	A	Outcome
T	H	H	Bob cleans
T	H	T	Chris cleans

There are only two equally likely outcomes. Therefore,

$$\implies P\{\text{Chris cleans}\} = \frac{1}{2}$$

4. (10 points) The lifetime of batteries are independent exponential random variables, each with parameter λ. A flashlight needs two batteries to function. Suppose you have a flashlight and a stockpile of n batteries. Let T be the total time that that flashlight can operate. What is the distribution of T?

Solution: Two batteries are inserted into the flashlight. The time to failure is given by the series connection of two independent components with exponential lifetimes. Therefore, the time to the first battery failure is $T_1 \sim \text{Exp}(2\lambda)$.

The dead battery is removed from the flashlight and a new one is inserted. By the memoryless property, the time to failure of the battery that remained in the flashlight remains exponential with rate λ. So, the time to the second battery failure is $T_2 \sim \text{Exp}(2\lambda)$ and is independent of T_1.

This process (remove a dead battery and replace with a fresh one from the stockpile) continues for a total of $n-1$ iterations.

Therefore, $T = \sum_{i=1}^{n-1} T_i \sim \text{Erlang}(n-1, 2\lambda)$.

5. (10 points) Consider the following cooling system for a server rack composed of two subsystems S_1 and S_2. Subsystem S_1 is composed of two coolers A and B, and subsystem S_1 fails if either A or B fail. Subsystem S_2 has a single cooler C which acts as a backup of subsystem S_1 and will be powered up only if subsystem S_1 fails. The overall system fails if both S_1 and S_2 fail.

   Assume failure detection and subsystem switching is perfect. Model the lifetimes of the coolers A, B, and C with three independent random variables $X_1 \sim \text{Exponential}(\lambda)$, $X_2 \sim \text{Exponential}(\lambda)$, and $X_3 \sim \text{Exponential}(3\lambda)$, respectively.

   (a) (5 points) What is the lifetime of the overall system?

   Solution: The lifetime of S_1 is the series connection of two iid components with exponential lifetimes. Therefore, the lifetime of S_1 is $F_{S_1}(t) = 1 - e^{-2\lambda}$. The lifetime of S_2 is $F_{S_2}(t) = 1 - e^{-3\lambda}$. The lifetime of the overall system is the sum of independent subsystem lifetimes, each of which is exponential with different rates. Therefore, the system lifetime is hypoexponential with parameters 2λ and 3λ. Therefore:

$$F_S(t) = \begin{cases} 1 - \frac{3\lambda}{3\lambda - 2\lambda} e^{-2\lambda t} + \frac{2\lambda}{3\lambda - 2\lambda} e^{-3\lambda t}, & t \geq 0 \\ 0, & \text{otherwise} \end{cases}$$
$$= \begin{cases} 1 - 3e^{-2\lambda t} + 2e^{-3\lambda t}, & t \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

   (b) (5 points) What is the failure rate function of the overall system?

Solution: The system lifetime pdf is

$$f_S(t) = \begin{cases} \frac{6\lambda^2}{2\lambda-3\lambda}\left(e^{-3\lambda t} - e^{-2\lambda t}\right), & t \geq 0 \\ 0, & \text{otherwise} \end{cases}$$
$$= \begin{cases} 6\lambda\left(e^{-2\lambda t} - e^{-3\lambda t}\right), & t \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

Therefore, the overall system failure rate function is given by

$$h_S(t) = \frac{6\lambda\left(e^{-3\lambda} - e^{-2\lambda}\right)}{3e^{-3\lambda t} - 2e^{-2\lambda t}}, \ t \geq 0$$

6. (12 points) Suppose X and Y are zero-mean, unit-variance jointly Gaussian random variables with correlation coefficient $\rho = 0.8 = \frac{4}{5}$.

(a) (4 points) Determine $\mathrm{Var}(5X - 3Y)$.

Solution:

$$\begin{aligned} \mathrm{Var}(5X - 3Y) &= \mathrm{Cov}(5X - 3Y, 5X - 3Y) \\ &= 25\,\mathrm{Var}(X) - 2(15)\,\mathrm{Cov}(X, Y) + 9\,\mathrm{Var}(Y) \\ &= 25(1) - 2(15)\left(\frac{4}{5}\right) + 9 = 34 - 2(3)(4) = 10 \end{aligned}$$

(b) (4 points) Determine $\mathrm{P}\{(5X - 3Y)^2 \geq 40\}$.

Solution: Note that $Z = 5X - 3Y$ is Gaussian because X and Y are jointly Gaussian.

$$\mu_X = \mathrm{E}[5X - 3Y] = 0$$
$$\implies Z \sim N(0, 10)$$

$$\begin{aligned} \mathrm{P}\{Z^2 \geq 40\} &= \mathrm{P}\big\{\{Z \geq \sqrt{40}\} \cup \{Z \leq -\sqrt{40}\}\big\} \\ &= 2\,Q\left(\frac{\sqrt{40} - 0}{\sqrt{10}}\right) \\ &= 2\,Q(2) \end{aligned}$$

(c) (4 points) Determine $\mathrm{E}[Y \mid X = 2]$.

Solution:

$$\begin{aligned} \mathrm{E}[Y \mid X = 2] &= \mu_Y + \sigma_Y \rho_{X,Y} \left(\frac{X - \mu_X}{\sigma_X} \right) \\ &= (0) + (1) \left(\frac{4}{5} \right) \left(\frac{2 - 0}{1} \right) = \frac{8}{5} \end{aligned}$$

7. (8 points) Let $Y \sim \text{Uniform}(-1, 1)$ and $X = Y^2$.

(a) (2 points) What is the MMSE estimator of X given $Y = v$?

Solution: Clearly $E[X \mid Y = v] = v^2$.

(b) (2 points) What is the MMSE estimator of Y given $X = u$?

Solution: If we are given that $X = u$, then Y could have been equal to $\sqrt{u}$ or $-\sqrt{u}$. If $0 < Y < 1$, then we should conclude that $Y = \sqrt{u}$. If $-1 < Y < 0$, then we should conclude that $Y = -\sqrt{u}$. In other words, the distribution of Y conditioned on $X = u$ is the probability mass function

$$p_{Y|X=u}(v|u) = \begin{cases} \frac{1}{2}, & v = \sqrt{u} \\ \frac{1}{2}, & v = -\sqrt{u} \\ 0, & \text{otherwise} \end{cases}$$

Therefore, $E[Y \mid X = u] = 0$.

(c) (2 points) What is the LMMSE estimator of X given $Y = v$?

Solution: The covariance of X and Y is $\mathrm{Cov}(X, Y) = \mathrm{E}[Y^3] - \mathrm{E}[Y^2]\,\mathrm{E}[Y]$. Since the distribution of Y is symmetric and centered at 0, all odd moments are equal to $0 \Rightarrow \mathrm{E}[Y^3] = \mathrm{E}[Y] = 0 \Rightarrow \mathrm{Cov}(X, Y) = 0$. Therefore,

$$\widehat{\mathrm{E}}[X \mid Y = v] = \mu_X = \mathrm{E}[Y^2] = \mathrm{Var}(Y) + \mathrm{E}[Y]^2 = \frac{(b-a)^2}{12} + 0 = \frac{1}{3}$$

(d) (2 points) What is the LMMSE estimator of Y given $X = u$?

Solution: The MMSE estimator is already in the form of a linear estimator, i.e., $\widehat{\mathrm{E}}[Y \mid X = u] = 0$.

8. (12 points) Consider the game of darts. Suppose each thrown dart lands a distance X_i from the target center where X_i is a random variable with the Rayleigh distribution, i.e., the pdf of X_i is

$$f_{X_i}(x_i) = \begin{cases} \frac{x_i}{\theta} e^{-\frac{x_i^2}{2\theta}} & x_i \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

and $\theta > 0$ is the distribution scale parameter.

(a) (6 points) If dart throws are independent and you measure the result of n trials, i.e., $X_1 = x_1, \ldots, X_n = x_n$, what is the maximum likelihood estimate for the scale parameter θ?

Solution: The joint pdf is given by

$$f_{X_1,\ldots,X_n}(x_1,\ldots,x_n) = \begin{cases} \prod_{i=1}^n \frac{x_i}{\theta} e^{-\frac{x_i^2}{2\theta}} & x_1 \geq 0, \ldots, x_n \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

The likelihood function is then

$$l(\theta; x_1, \ldots, x_n) = \prod_{i=1}^n \frac{x_i}{\theta} e^{-\frac{x_i^2}{2\theta}}$$

and the log-likelihood function by

$$\begin{aligned} \ln l(\theta; x_1, \ldots, x_n) &= \sum_{i=1}^n \left[\ln \frac{x_i}{\theta} - \frac{x_i^2}{2\theta} \right] \\ &= \sum_{i=1}^n \ln x_i - n \ln \theta - \frac{1}{2\theta} \sum_{i=1}^n x_i^2 \end{aligned}$$

Differentiating the log-likelihood we find

$$\frac{d \ln l(\theta; x_1, \ldots, x_n)}{d\theta} = -\frac{n}{\theta} + \frac{1}{2\theta^2} \sum_{i=1}^n x_i^2$$

Finally, equating to 0 we obtain

$$\widehat{\theta}_{\text{ML}} = \frac{1}{2n} \sum_{i=1}^n x_i^2$$

(b) (6 points) Suppose the scale parameter for a professional player is equal to θ_p and is equal to $\theta_a = 10\theta_p$ for an amateur player. After n independent dart throws with $X_1 = x_1, \ldots, X_n = x_n$, what is the maximum likelihood decision rule to determine if a player is a professional or an amateur? Simplify your solution.

Solution: There are two hypotheses: H_0 the player is professional and H_1 the player is an amateur. We then have

$$f_0(x_1, \ldots, x_n) = \prod_{i=1}^{n} \frac{x_i}{\theta_p} \exp\left(-\frac{x_i^2}{2\theta_p}\right), \quad x_1 \geq 0, \ldots, x_n \geq 0$$

$$f_1(x_1, \ldots, x_n) = \prod_{i=1}^{n} \frac{x_i}{\theta_a} \exp\left(-\frac{x_i^2}{2\theta_a}\right), \quad x_1 \geq 0, \ldots, x_n \geq 0$$

$$= \prod_{i=1}^{n} \frac{x_i}{10\theta_p} \exp\left(-\frac{x_i^2}{20\theta_p}\right)$$

The likelihood ratio is given by

$$\Lambda(x_1, \ldots, x_n) = \frac{f_1(x_1, \ldots, x_n)}{f_0(x_1, \ldots, x_n)}$$

$$= \frac{1}{10} \prod_{i=1}^{n} \exp\left(-\frac{x_i^2}{20\theta_p} + \frac{x_i^2}{2\theta_p}\right)$$

$$= \frac{1}{10} \prod_{i=1}^{n} \exp\left(\frac{9x_i^2}{20\theta_p}\right)$$

Finally, the maximum likelihood decision rule is

$$\Lambda(x_1, \ldots, x_n) \underset{H_0}{\overset{H_1}{\gtrless}} 1$$

$$\sum_{i=1}^{n} x_i^2 \underset{H_0}{\overset{H_1}{\gtrless}} \frac{20\theta_p \ln 10}{9}$$