Numerical Python

CS*101* `lec15`

Solving Equations

2019-11-06

Announcements

quiz: `quiz15` due on Thurs 07/11

lab: `lab` on Fri 08/11

hw: `hw07` due today

`exam02` on 13/11

Roadmap

- Week 1 (9/9): L0 Intro; L1 PY - Intro
- Week 2 (16/9): L2 PY – Data& Bool; L3 PY – if & Loops
- Week 3 (23/9): L4 PY - Lists; L5 PY - Funcs
- 30/9: L6 PY - Files; ✗
- 9/10: L7 PY - Muta
- 12/10: Exam 01
- Week 6 (14/10): L8 PY - Dict; L9 PY - IO
- Week 7 (21/10): L10 PY - Numpy; L11 PY - Plot
- Week 8 (28/10): L12 PY - State; L13 PY - Rand
- Week 9 (4/11): L14 PY – De-bug; L15 PY - Eqn
- Week 10 (11/11): L16 PYA I; Exam 02
- Week 11 (18/11): L17 PY - Opt II; L18 PY - Mod
- Week 12 (25/11): L19 PY – Sym I; L20 PY – Sym II
- Week 13 (2/12): L21 PY – Sym III; L22 ML - Intro
- Week 14 (9/12): L23 ML - IO; L24 ML - Poly
- Week 15 (16/12): L25 ML - Stats; L26 ML - Fit
- Week 16 (23/12): Review
- Week 17 (30/12): Finals

Objectives

A. Represent and solve equations in an efficient manner. => Similar to **lec on Random and Numpy**
B. Locate a function’s zeroes using a graphical method or Newton’s method. => **lec on plotting**
C. Locate a function’s minima using a graphical method or ‘scipy.optimize.minimize‘.

Question

```
x = np.ones( 10 )
for i in range( 10 ):
    try:
        ???
    except:
        print( 'Error on step %d.'%i )
        continue
```

Which of the following candidates for `???` would *not* produce an error message?

A `x += x[ i+1 ]`

B `x[ i ] /= 0`

C `x[ -i-1 ] = sum( x[ :i ] )`

D `x[ 10-i ] = sum( x[ :i ] )`

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Which of the following candidates for `???` would *not* produce any error message?

A `x += x[ i+1 ]` index error

B `x[ i ] /= 0` ⋆(surprise! `numpy` can handle)

C `x[ -i-1 ] = sum( x[ :i ] )` ⋆

D `x[ 10-i ] = sum( x[ :i ] )` index error

Equations

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C. Write as a series

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1. In other words, we convert the equation into something that can be calculated. We want *numbers* out of them.
2. Many times we represent the function as a pair of arrays, `x` and `y` (like for plotting).
3. We can also represent equations using symbols from the library `sympy`, (later lectures).

Equations

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$$y = a\sin^3 x + b\sin^2 x + c\sin x + d$$

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A.  y = a*sin(x)**3 + b*sin(x)**2 + c*sin(x) + d

B.  t = sin(x)
y = a*t**3 + b*t**2 + c*t + d
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```

The first way takes three times longer!

$\sin$ is calculated every single time it is used.

Equations

What about calculating π? Which is faster?

A. The Monte Carlo method?

B. Series solution?

$$\frac{\pi}{4} = +1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \frac{1}{15}, \ldots$$

π using Monte Carlo

```
import numpy.random as npr
def mc_pi( n ):
  xy = npr.rand( n,2 ) * 2 - 1
  n_circle = 0
  for pair in xy:
    ........
  return estimate
```

π using Series summation

```
def series_pi( n ):
  result = 0
  for k in range( 1,n ):
    term = ( ( -1 ) ** ( k+1 ) ) / ( 2 * k - 1 )
    result += term
  return result*4
```

Equations

Which way is more efficient computationally?

Equations

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The series solution is much better, and other better ways may exist.

We can quantify this if we can compare algorithm.

How to quantify?

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or

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  - ii. `» code` = śome code but as a string ´ `» timeit.timeit(code, number=10000)` or
  - iii. `» timeit.timeit(code, setup = optional, number=10000)`
- `number` = the number of times to run to get an average time
- `setup` = setup python before running `code`. e.g., setup = import math

Code performance

```
import timeit
```

B. Jupyter notebook: `%timeit codeJupyter` (this is easiest)

`codeJupyter` is just your `def` function

These commands run your code many times and return an average time to completion.

```
%timeit  mc_pi( 1e5 )
%timeit  series_pi( 1e5 )
```

Code performance example

Jupyter:

```
def fib_a( n ):
    sqrt_5 = 5**0.5;
    p = ( 1 + sqrt_5 ) / 2;
    q = 1 / p;
    return int( (p**n + q**n) / sqrt_5 + 0.5 )

%timeit -n 10 fib_a(50)
```

`-n 10` means run 10 times

Fibonacci sequence example

$$F_n = F_{n-1} + F_{n-2} \qquad F_1 = F_2 = 1$$

$$1\,,1\,,2\,,3\,,5\,,8\,,13\,,21\,,34\,,55\,,...$$

Fibonacci sequence example

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$$1\,,1\,,2\,,3\,,5\,,8\,,13\,,21\,,34\,,55\,,...$$

The closed-form formula for the nth Fibonacci term is:

$$F_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{2}{1+\sqrt{5}}\right)^n}{\sqrt{5}+\frac{1}{2}}$$

Analytical Fibonacci

```
def fib_a( n ):
    sqrt_5 = 5**0.5;
    p = ( 1 + sqrt_5 ) / 2;
    q = 1 / p;
    return int( (p**n + q**n) / sqrt_5 + 0.5 )
```

Recursive Fibonacci

```
def fib_r( n ):
    if n == 1 or n == 2:
        return 1
    else:
        return fib_r( n-1 ) + fib_r( n-2 )
```

Comparison

```
%timeit fib_a( 12 )
%timeit fib_r( 12 )
```

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%timeit fib_a( 12 )
%timeit fib_r( 12 )
```

On my machine, `fib_a` is 55 $\times$ faster than `fib_r` for `n = 12`.

Will this performance get better or worse for larger `n`?

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$$\begin{aligned} \exp(-x) &= 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + ... \\ &= \frac{x^0}{0!} - \frac{x^1}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + ... \end{aligned}$$

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This series is well-behaved, *but*...

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Intermediate terms can behave like:

if x = 10,

$$\frac{10^5}{5!} = \frac{100,000}{120} = 833.333$$

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if x = 10,

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or

$$\frac{10^{12}}{12!} = \frac{1,000,000,000,000}{479,001,600} = 2,087.675$$

Very large numbers result, leading to inefficient calculation and possible numerical error.

Alternating negative terms will also lead to numerical errors.

So what can we do?

Equations - series

To break a big number to a combination of smaller number.
In this case, use

$$e^x = \frac{1}{e^{-x}}$$

$$e^x = \left(e^{\frac{x}{n}}\right)^n$$

Original:

$$e^{12} = \frac{12^0}{0!} - \frac{12^1}{1!} + \frac{12^2}{2!} - \frac{12^3}{3!} ... + \frac{12^{10}}{12!} + ..$$

Improved:

$$e^{12} = \left(e^{\frac{12}{4}}\right)^4 = e^3 * e^3 * e^3 * e^3$$

$$e^3 = \frac{3^0}{0!} - \frac{3^1}{1!} + \frac{3^2}{2!} - \frac{3^3}{3!} ... + \frac{3^{10}}{12!} + ...$$

Second one has smaller numbers to divide

Question

Suppose that you wish to evaluate the function:

$$t(x) = a\exp(3x) + b\exp(2x) + c\exp(x).$$

A
```
t = a*exp(3*x) + b*exp(2*x) + c*exp(x)
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B
```
z = exp(x)
t = a*z**3 + b*z**2 + c*z + d
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Suppose that you wish to evaluate the function:

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```
z = exp(x)
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```

⋆⋆⋆

Solving Equations in x

Solving eqns

Let's consider how to find a specific solution to an equation, a value of x for which $f(x)$ has a desired property.

Methods:

Solving eqns

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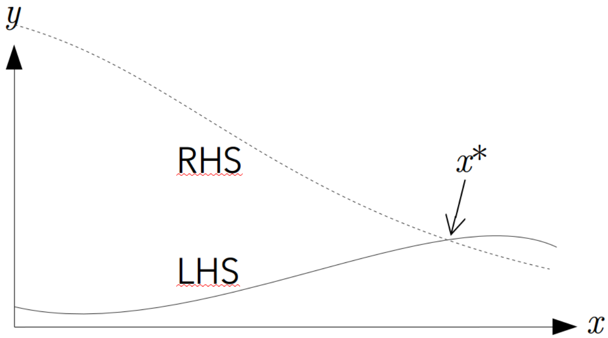
Methods:

A. Plot LHS == RHS
B. Newton's method or variant
C. Use scipy.optimize
... (more)

Solving eqns - Plot

The easiest way is to plot LHS v. RHS and find the crossover point:



Solving eqns - Plot

$$x^2 + 5x - (2x^2 - 3) = -2x^2 - x$$

```
x**2 + 5*x - (2*x**2 - 3) == -2*x**2 - x
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```

```
x = np.linspace( -10,10,1001 )
lhs = x**2 + 5*x - (2*x**2 - 3)
rhs = -2*x**2 - x
plt.plot( x,lhs,'r', x,rhs,'b' )
plt.plot( x,lhs-rhs,'g' )
```

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```

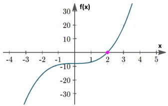
This works, but we need something better than eyeballing it.

Solving eqns - Newton's method

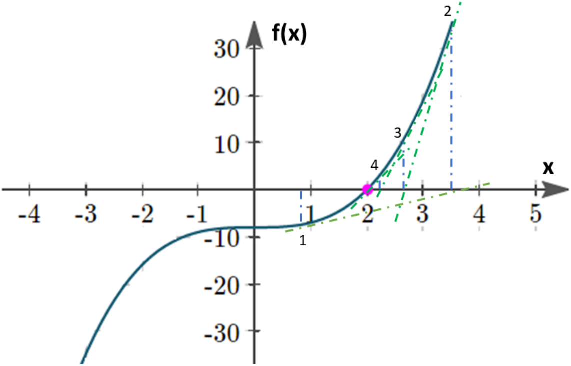
Newton's method uses the function and its derivative to locate the x-value of the zero, x^*.

The trick, of course, is that you need $f'(x) = \frac{d[f(x)]}{dx}$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$



$\rightarrow$



Solving eqns - Newton’s method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

```
def dfdx( f,x,h=1e-3 ):
    return ( f( x+h ) - f( x ) ) / h

def newton( f,x0,tol=1e-3 ):
    d = abs( 0 - f( x0 ) )
    while d > tol:
        x0 = x0 - f( x0 ) / dfdx( f,x0 )
        d = abs( 0 - f( x0 ) )
    return ( x0,f( x0 ) )
```

Questions

For

$$\cos x + 2 = x^3 - x^2$$

What are the parameters needed for `newton( f,x0,tol=1e-3 )` to work?

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What are the parameters needed for `newton( f,x0,tol=1e-3 )` to work?

```
def f(x):
    import numpy as np
    return (( np.cos( x ) + 2 ) - ( x**3 - x**2 ))

x0 = any number

newton( f, x0, tol=1e-3 )
```

Solving eqns - scipy.optimize

`import scipy.optimize`

There is a ready-made Newton's method in scipy.optimize

`> scipy.optimize.newton( f,x0 )`

We can also find minima using

`> scipy.optimize.fmin( f,x0 )`.

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```

This requires you to be clever in preparing `f`, you may have to manipulate your function.

Solving eqns - scipy.optimize

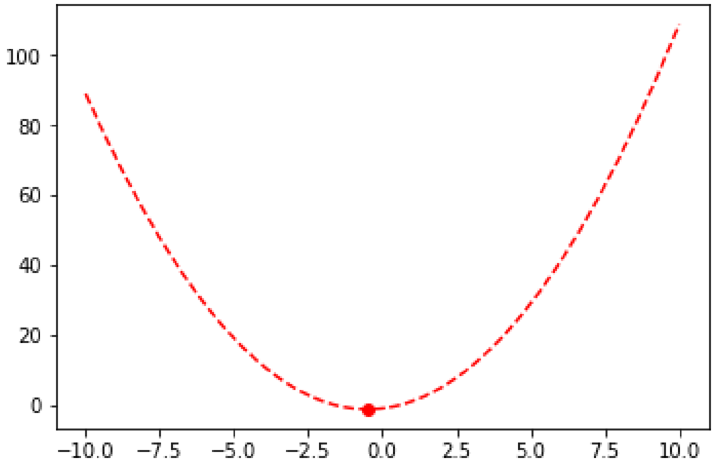
```
import matplotlib.pyplot as plt
import numpy as np
import scipy.optimize

def f( x ):
    return x**2 + x - 1

x = np.linspace( -10,10,1000 )
xstar = scipy.optimize.fmin( f,x0=3 )

plt.plot( x,f( x ),'r--', xstar,f( xstar ),'ro' )
plt.show()
```

Solving eqns - scipy.optimize



Optimization (Preview)

Optimization

On vacation, you purchased a range of n souvenirs of varying weight and value. When it comes time to pack, you find that your bag has a weight limit of 22 kg. What is the best set of items to take on the flight?

Summary

A. Choose the correct way to represent equations

    More function calls $\rightarrow$ slower

    Simple codes are generally faster

B. `import timeit` to time commands

C. Solution methods

    Plotting graphs to find solutions to equations $\rightarrow$ intersections

    Newton's method

    `import scipy.optimize as sco`

    `sco.newton(...)`

    `sco.fmin(...)`

    `sco.minimize(...)` more powerful but complicated than `sco.fmin(...)`